MAC 2233-07B
Final Exam Review
Name: November 27, 2019

Chapter 1: Algebra Review

1.2 Operations on Functions

Remember, the *composition* of two functions $f(x)$ and $g(x)$ is denoted by $f(g(x))$. Here, the *output* of g becomes the *input* of f. In other words, every x that shows up in the formula for $f(x)$ gets replaced by $g(x)$.

Example:

Suppose $f(x) = \sqrt{x}$ and $g(x) = 6x + 1$. If we compose the two, we get

$$f\left(g(x)\right) = \sqrt{6x+1}$$

However, we could compose them the other way, making g the outside function, in which we get

$$g\left(f(x)\right) = 6\sqrt{x} + 1.$$

Transformations

Given a function $f(x)$, recall that the function $g(x) = f(x) + k$ corresponds to a vertical shift of $f(x)$ upwards of k units. The function $h(x) = f(x + k)$ corresponds to a horizontal shift of $f(x)$ k units to the *left*.
Given a function $f(x)$, the new function $g(x) = -f(x)$ is a vertical reflection of $f(x)$, i.e. a reflection about the x-axis. The new function $h(x) = f(-x)$ is a horizontal reflection of $f(x)$, i.e. a reflection about the y-axis. I don't have any pictures to attach for this, so make sure to do the problems on MyOpenMath.

1.3 Linear Functions

A linear function is a function whose graph is a line. More specifically, it is given by the formula $f(x) = mx + b$ or $f(x) = b + mx$, where m is the slope of the line, and b is where the function crosses the y-axis, i.e. the point $(0, b)$ is the y-intercept.

Slope Formula

Given two points (x_1, y_1) and (x_2, y_2), the slope of the line that passes through these points is given by

$$m = \frac{y_2 - y_1}{x_2 - x_2}$$

1.4 Exponents

Remember the following exponent rules:

$x^a \cdot x^b = x^{a+b}$	$\dfrac{x^a}{x^b} = x^{a-b}$	$(x^a)^b = x^{ab}$
$(xy)^a = x^a y^a$	$\left(\dfrac{x}{y}\right)^a = \dfrac{x^a}{y^a}$	$x^0 = 1$ (unless $x = 0$)

1.5 Quadratics

A quadratic function is a function written in the form $f(x) = ax^2 + bx + c$. This is sometimes called standard form. Another way to express a quadratic is using the vertex form, $f(x) = a(x-h)^2 + k$, where the vertex of the quadratic is (h, k).

Quadratic Formula

If you can't factor the quadratic, you can find the horizontal intercepts (if any) by using the equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.6 Polynomials and Rational Functions

A polynomial is a function of the form $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. The *degree* of the polynomial is the highest power of x that appears. The *leading coefficient* of the polynomial is the coefficient of the highest power of x.

Example:

The following function is a polynomial:

$$f(x) = 1 + 2x + 3x^2 + 5x^5 + 2x^6$$

Here, the degree is 6 and the leading coefficient is 2.

Rational Fuctions

A rational function is a function that is the quotient of two polynomials, e.g. $f(x) = \dfrac{2x+1}{3x^2-2}$.

Short Run Behavior

The short run behavior of a function is when the function is approaching infinity as x approaches some value. This behavior creates a *vertical asymptote*.
For rational functions, the vertical asymptote is given by the x-value when the denominator is 0 and the numerator is not 0.

Long Run Behavior

The long run behavior of a function is when the function is approaching a certain value as x itself approaches infinity (grows very large). This behavior creates a *horizontal asymptote*.
For rational functions, the horizontal asymptote can be found by using the following rules:

a. Degree of denominator > degree of numerator: horizontal asymptote at $y = 0$.

b. Degree of denominator < degree of numerator: no horizontal asymptote

c. Degree of denominator = degree of numerator: horizontal asymptote at ratio of leading coefficients.

Example:

Consider the following rational function:

$$f(x) = \frac{3x^2}{2x^2 + x + 1}$$

Since the degree on top is the same as the degree on the bottom, the horizontal asymptote occurs at the ratio of the leading coefficients. The coefficient on top is 3 and the coefficient on the bottom is 2, so the horizontal asymptote is the line $y = \frac{3}{2}$.

1.7 Exponential Functions

An exponential function is given by the equation $f(x) = a(1 + r)^x$, where a is the initial value and r is the growth/decay rate given as a decimal. It can be written in the form $f(x) = a \cdot b^x$, where $b = 1 + r$.

Continuous Growth

Sometimes growth is continuous (not compounded at discrete intervals). If this is the case, continuous growth is given by the function $f(x) = ae^{rx}$ where a is the initial value, e is Euler's number ≈ 2.718282, and r is the growth rate.
Note: refer to page 63 of the textbook on Canvas to look at graphs of exponential equations.

1.8 Logarithmic Functions

The logarithm (base b) function is expressed as $f(x) = \log_b(x)$. When trying to understand this function, recall that the statement $\log_b(a) = x$ is equivalent to the statement $b^x = a$. We denote the common logarithm as $\log(x)$, which has base 10. The natural logarithm is $\ln(x)$, which has base e.

Example:

We know that $4^3 = 64$. Using the definition, we can evaluate $\log_4(64)$:

$$4^3 = 64$$
$$\Updownarrow$$
$$\log_4(64) = 3$$

Graphical Features

Refer to page 70 to look at the behavior/graphs of the logarithm. The most important feature of the graph is the domain. For the function, $f(x) = \log_b(x)$, the domain is $(0, \infty)$ (can you think of why that may be?).

Chapter 2: The Derivative

2.2 Limits and Continuity

The idea behind a limit is that we want to describe the behavior of a function near a certain point, but not specifically at the point. We denote the "limit of $f(x)$ as x approaches a as:

$$\lim_{x \to a} f(x)$$

I don't have any graphs I can attach, so practice the limit problems that show up in MyOpenMath. The reason the limit is important is so that we can define the derivative.

2.3 The Derivative

The derivative of a function $f(x)$ at a point a is defined to be the *instantaneous* rate of change of the function at the point a. We express the derivative with the following two equivalent notations:

$$\frac{df}{dx} \Leftrightarrow f'(x)$$

It's given by the following formula (which you do NOT have to memorize, just understand it's meaning):

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$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2.5 Derivatives of Formulas

Remember the following rules for differentiation:

The Product Rule

Rule for taking the derivative of the product of two functions:

$$(f(x) \cdot g(x))' = f'(x) \cdot g(x) + g'(x) \cdot f(x)$$

Example:

Let's find the derivative of

$$h(x) = 2xe^x.$$

To use the product rule, we need to split it it up into two functions $f(x)$ and $g(x)$ and find their derivatives:

$$f(x) = 2x \qquad g(x) = e^x$$

$$f'(x) = 2 \qquad g'(x) = e^x$$

We can now apply the product rule:

$$h'(x) = 2e^x + 2xe^x$$

The Quotient Rule

Rule for taking the derivative of the quotient of two functions:

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - g'(x)f(x)}{[g(x)]^2}$$

Example:

Let's find the derivative of

$$h(x) = \frac{\ln(x)}{5x}.$$

Much like the product rule, in order to use the quotient rule, we need to split $h(x)$ up into two functions $f(x)$ and $g(x)$ and find their derivatives:

$$f(x) = \ln(x) \qquad g(x) = 5x$$

$$f'(x) = \frac{1}{x} \qquad g'(x) = 5$$

We can now apply the quotient rule:

$$h'(x) = \frac{\frac{1}{x} \cdot 5x - 5\ln(x)}{(5x)^2} = \frac{1 - \ln(x)}{5x^2}$$

(Note: I simplified in the last step)

The Chain Rule

Rule for taking the derivative of the *composition* of two functions (remember f of g):

$$[f(g(x))]' = f'(g(x)) \cdot g'(x)$$

Example:

Suppose we want to take the derivative of

$$h(x) = \sqrt{3 + 5x}.$$

You might notice that the other rules won't apply here, so we hope that the chain rule will work. It seems like there's an obvious inside function, the one that's "inside" the square root. Hence, we can split $h(x)$ up into $f(x)$ and $g(x)$ and find their derivatives as follows:

$$f(x) = \sqrt{x} \qquad g(x) = 3 + 5x$$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad g'(x) = 5$$

Hence, we can now apply the chain rule:

$$h'(x) = \frac{1}{2\sqrt{3 + 5x}} \cdot 5 = \frac{5}{2\sqrt{3 + 5x}}.$$

Applications

Given a revenue function $R(x)$ (revenue = price $\times$ quantity), the derivative $R'(x)$ is called the **marginal revenue** function. Similarly, given a cost function $C(x)$, the derivative $C'(x)$ is called the **marginal cost** function.
Given a position function $s(t)$, the amount of distance travelled after t units of time, the **velocity** is given by the first derivative $s'(t)$ and the **acceleration** is given by the second derivative $s''(t)$.
While you definitely don't need to know this unless you want to go on Jeopardy or something, the 3$^{\text{rd}}$-6$^{\text{th}}$ derivatives of position are jerk, snap, crackle, and pop.

2.6 Second Derivative and Concavity

For a given function $f(x)$, we can find the second derivative of f'' by taking the derivative of $f'(x)$, i.e. taking the derivative twice in a row. We get some nice properties with the second derivative.
If $f''(x) > 0$, then f is **concave up** at x. If $f''(x) < 0$, then f is **concave down**. We say f has an **inflection point** at x if the concavity shifts at x, i.e. the second derivative changes sign. Hence, if f has an inflection point at x, then either $f''(x) = 0$ or $f''(x)$ is undefined. To find inflection points, set the second derivative equal to zero and solve for x. Once you do this, check to make sure x is actually an inflection point by plugging in values around x and making sure the concavity does indeed shift.

Example:

We know the function $f(x) = -3x^2 + 1$ has an upside-down parabolic shape, so we expect it to be concave down. Let's check it:

$$f(x) = -3x^2 + 1$$
$$\Downarrow$$
$$f'(x) = -6x$$
$$\Downarrow$$
$$f''(x) = -6$$

Since $f''(x) < 0$ for any value of x you choose, the function is concave down.

2.7 Optimization

Optimization is concerned with finding the maximum and minimum value of a function. We say that a function f has a **critical point** at x if $f'(x) = 0$ or $f'(x)$ is undefined. Now if f has a critical point at x and $f''(x) < 0$, then f has a **local maximum** at x. Similarly, if f has a critical point at x and $f''(x) > 0$, then f has a **local minimum** at x.

How to solve an optimization problem: First, take the derivative of $f(x)$ and set it equal to zero to find all of your critical points. After solving, use the second derivative test to label your critical points as a maximum or a minimum (or neither!). At the end of the day, you're essentially just looking to see what the maximum/minimum value of the function is, so if you get several maximums/minimums from the previous steps, just plug them back into the function to see which one is bigger/smaller. Finally, when you are done all of that, plug in the endpoints too if there are any (we do this because it is technically possible that the endpoints are bigger/smaller than the maximum/minimum you found from the derivatives) and voila! You've got yourself some nice extrema.

Example:

Let's find the local maximum/minimum values of the function $f(x) = x^3 + 4x^2 + 4x - 2$. We start by taking the first derivative:

$$f'(x) = 3x^2 + 8x + 4$$

Now set it equal to zero and solve:

$$\begin{gathered} 3x^2 + 8x + 4 = 0 \\ \Downarrow \\ (x + 2)(3x + 2) = 0 \\ \Downarrow \\ x = -2 \text{ or } x = -\frac{2}{3} \end{gathered}$$

We now need to find which one is a maximum and which one is a minimum. We use the second derivative:

$$f''(x) = 6x + 8.$$

Plugging the critical points in gives us $f''(-2) = -4$ and $f''\left(-\frac{2}{3}\right) = 4$. Hence, f has a maximum at $x = -2$ and a minimum at $x = -\frac{2}{3}$ (If you're confused about this, think about how concavity relates to maxima/minima).
Finally, plug in the x-values to get the maximum and minimum values of the function.
Maximum: $f(-2) = -2$
Minimum: $f\left(-\frac{2}{3}\right) = -3.185$

2.10 Other Applications

Give a demand function that relates the quantity q to the price p, the *elasticity* of demand is given by

$$E = \left| \frac{p}{q} \cdot \frac{dq}{dp} \right|$$

In cases where the demand function is explicitly given as $q = D(p)$, it's equivalent to think of elasticity with the equation

$$E = \left| \frac{p}{D(p)} \cdot D'(p) \right|$$

The interpretation of elasticity: "if price increases by 1%, then demand decreases by E%". When $E < 1$, we say that demand is **inelastic** and increasing prices will increase revenue. When $E > 1$, we say that demand is **elastic** and increasing prices will decrease revenue. When $E = 1$, we say that demand is **unitary**. Note that $E = 1$ at critical points of the revenue function.

2.11 Implicit Differentiation and Related Rates

The idea of related rates and implicit differentiation is that sometimes we need to take derivatives with respect to multiple variables listed in the problem. The following formula helps with that:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

If you need a further explanation on this, take a look at the attachment I posted to Canvas before Exam 2.

3.2 The Fundamental Theorem and Antidifferentiation

The area under the curve of a given function can be calculated by analyzing the anti-derivative. This can be summed up with the following essential formula:

The Fundamental Theorem of Calculus

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

3.3 Antiderivatives of Formulas

Here are some handy integration formulas that makes calculating the anti-derivative a little bit easier:

The Power Rule

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C$$

for all n except the special case $n = -1$.

Natural Logarithm

The special case of the above formula, when $n = -1$, is as follows:

$$\int x^{-1}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C$$

Exponential Functions

$$\int e^x\,dx = e^x + C$$
$$\int a^x\,dx = \frac{a^x}{\ln a} + C$$

3.4 Substitution

Another common integration trick is that of u-substitution, given by the following formula:

$$\int f\left(g(x)\right)\,dx = \int f(u)\,du$$

where we set $u = g(x)$ and $du = g'(x)\,dx$. The idea behind u-substitution is to take the "messy" part of the integral and make it one step easier by cleaning it up with u. There is no exact method that can be used to solve a u-substitution problem, it just takes practice. Trial and error is not necessarily a bad thing when calculating integrals.

Example:

Suppose we want to evaluate the following integral:

$$\int \frac{2x}{\sqrt{x^2+10}}\,dx$$

Unfortunately, none of the standard integral rules can apply here, so we have to do something else. The only other trick we know how to do is u-substitution, so that better work or else we're stuck. The trick behind u-substitution is to locate the most difficult aspect of the integral, in this case I would say it is the term $\sqrt{x^2+10}$. Then, choose u to make it "one-step easier." In other words, instead of setting $u = \sqrt{x^2+10}$, just set $u = x^2 + 10$, leaving the square root out of this. When we do that, we get the following integral:

$$\int \frac{2x}{\sqrt{u}}\,dx$$

We now have an even bigger problem than the one we started with since there are now TWO variables inside our integral. However, we can take care of that:

$$u = x^2 + 10$$
$$\Updownarrow$$
$$\frac{du}{dx} = 2x$$
$$\Updownarrow$$
$$du = 2x\,dx$$

Hence, we can replace the term $2x\,dx$ inside our integral with du. We can then solve:

$$\int \frac{2x}{\sqrt{u}}\,dx = \int \frac{1}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}}\,du = 2\sqrt{u} + C$$

Now, you may be tempted to call this the answer, but there is still one more step. We started with a function in terms of x, and we ended up with a function in terms of u, which is illegal. Hence, we have to replace u with its value $x^2 + 10$:

$$2\sqrt{u} + C \implies 2\sqrt{x^2+10} + C$$

So, our final answer is $2\sqrt{x^2+10} + C$.

Area and Average Value

If one wants to calculate the area that is bounded by the graph between two curves $f(x)$ and $g(x)$, the following formula applies:

$$A(x) = \int_a^b f(x) - g(x)\,dx$$

In general, you need to make sure that $f(x)$ is the function on top, and $g(x)$ is the function on the bottom. If the top and bottom functions switch inside the interval, you may have to split it up into two different integrals.

Average Value

The average value of a function $f(x)$ on the interval $[a, b]$ is given by the following formula:

$$H = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Example:

Suppose we want to find the average value of the function $f(x) = 4x^3$ on the interval $[1, 3]$. We simply apply the formula:

$$H = \frac{1}{3-1}\int_1^3 4x^3\,dx = \frac{1}{2}x^4\Big|_1^3 = \frac{1}{2}(3^4) - \frac{1}{2}(1^4) = 40.$$

Applications to Business

Given a demand function $D(q)$, a supply function $S(q)$, and an equilibrium point (q^*, p^*), we have the following two formulas:

Consumer Surplus

$$\int_0^{q*} D(q)\,dq - p^*q^*$$

Producer Surplus

$$p^*q^* - \int_0^{q^*} S(q)\,dq$$

The sum of consumer surplus and producer surplus is the **total gains from trade**.
The best problems to practice for this topic is from the Homework 3 review, email me if you have any questions. I'll provide one example though:

Example:

Suppose the demand for a given product is given by the equation $D(q) = 10 - 3q$ and the supply for the product is given by the equation $S(q) = q + 2$. Let's try to find the Consumer Surplus.

Before we can do any of the very fun and exciting calculus, we must first find the equilibrium point (q^*, p^*). To do this, just set the supply and demand equal to each other and solve:

$$S(q) = D(q)$$
$$\Downarrow$$
$$q + 2 = 10 - 3q$$
$$\Downarrow$$
$$4q = 8$$
$$\Downarrow$$
$$q = 2$$

Thus, $q^* = 2$ is the equilibrium quantity. To find the equilibrium price, just plug in q^* to either $D(q)$ or $S(q)$. This gives us the equilibrium point $(2, 4)$. We can now find the consumer surplus:

$$\int_0^{q*} D(q)\, dq - p^*q^* = \int_0^2 10 - 3q\, dq - 8 = 14 - 8 = 6.$$

Continuous Income Stream

Suppose money can earn interest at an annual interest rate of r, compounded continuously. If $F(t)$ is a continuous income function that applies between the year 0 and the year T, then the present value of the income stream is

$$PV = \int_0^T F(t)e^{-rt}$$

and the future value is

$$FV = PVe^{rt}.$$

This concept can be a little tricky to understand. The idea is this: if I am making money continuously over the next T years, and I assume that I am investing my earnings as soon as they come in, what is the flat amount of money that I have to invest *right now* such that the two balances will be the same after T years. It's probably easiest to see it with an example.

Example:

Suppose a company wishes to invest in a machine that costs \$60,000. The company estimates that the machine will make \$6,500 per year for a duration of 10 years. If money earns 2% annually, is this a wise purchase?

On first glance, it seems like this would be a smart purchase since \$6,500 for 10 years equates to a total of \$65,000 dollars, which is more than the machine costs. However, the idea behind present value is that \$60,000 right now could possibly be worth more than \$65,000 in 10 years from now, since we can invest our money during that time. See, if we invest \$60,000 right now over the next 10 years at 2% annual interest, we would have

$$A = Pe^{rt} = 60,000e^{0.02(10)} = \$73,284.17$$

after 10 years. Just by looking at this number, we see that the \$65,000 that the machine makes is less than the amount we would get by simply investing the \$60,000 the machine costs. However, it is a little more complicated than that.

We now get to the beauty of the present value formula. Since the machine is earning money continuously, we can assume that the company will invest the earnings they make continuously as well, at a 2% interest rate. In other words, the machine is actually making more than the \$65,000 we originally thought because every time it makes money, we'll be investing it. While it's difficult to calculate exactly how much the machine is going to make, we can in fact calculate exactly how much is to be invested *right now* to match the earnings. That's given by the present value formula:

$$PV = \int_0^{10} 6500e^{-0.02t}\,dt = \$58,912.50$$

Hence, the amount of money that the machine will earn over the next 10 years (taking into account investments) is equal to the amount of money that would be earned by investing \$58,912.50 right now for the next 10 years. Since the machine costs \$60,000, it would be smarter to just invest the money instead of purchasing the machine.

So the answer is no, it is not a wise purchase. While we're done with the problem, it still might be interesting to calculate exactly how much the machine is making us. We don't have a formula to help us calculate explicitly what the machine is earning with investments taken into account, but we do know that the value will be the same as investing \$58,912.50 at 2% annual interest for the next 10 years. Hence, the machine will make us exactly

$$A = Pe^{rt} = 58,912.50e^{0.02(10)} = \$71,955.89$$

You might notice that this amount is indeed smaller than the amount we would get from investing the \$60,000 to begin with. This just proves that what we thought was correct: the machine is not a smart investment.

Thanks for coming to my TED talk.