MAP 3305-002 Final Exam Review Packet

This packet is a comprehensive study guide for the final exam. Any of the material that is covered in this packet may (and probably will) appear on the final exam. More importantly, sections which are not covered in this packet *will not* be on the final exam.

Topics that will be Covered	**Topics that will NOT be Covered**
2.1: Integrating Factor	2.6: Exact Equations
2.2: Separable Equations	3.4: Reduction of Order
2.3: Tank/Mixing Problems	6.1: Integral Def. of Laplace Transform
3.1, 3.3, 3.4: Characteristic Equations	6.5: Impulse Functions
3.5: Undetermined Coefficients	
4.2: Higher Order Equations	
6.2: IVPs with Laplace	
6.3: Step Functions	
6.4: IVPs with Step Functions	

The idea of this review is to *supplement* your homework and give you a brief summary of everything we've covered this semester that will be on the final. At the end of each section, I've highlighted the assigned homework problems for that section and added a few extra practice problems.

oh and also there are solutions in the back :)

Chapter 2: First Order Differential Equations

2.1 Linear Equations; Method of Integrating Factors

The method of integrating factors can be used to solve first-order *linear* differential equations.

Linear Differential Equation: $y' + p(t)y = g(t)$

Aptly named, this method is performed by introducing a new function $\mu(t)$ called the *integrating factor*.

Integrating Factor: $\mu(t) = e^{\int p(t)\,dt}$

The following is a step-by-step method to solve integrating factor problems:

Step 1: Make sure equation is in the correct form.

$$y' + p(t)y = g(t)$$

Step 2: Calculate integrating factor.

$$\mu(t) = e^{\int p(t)\,dt}$$

Step 3: Multiply integrating factor to both sides.

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

Step 4: Use reverse product rule to simplify left-hand side.

$$\frac{d}{dt}\left(\mu(t)y\right) = \mu(t)g(t)$$

Step 5: Integrate both sides.

$$\int \frac{d}{dt}\left(\mu(t)y\right)\,dt = \int \mu(t)g(t)\,dt \qquad \Longrightarrow \qquad \mu(t)y = \int \mu(t)g(t)\,dt + C$$

Step 6: Divide both sides by $\mu(t)$ to get general solution:

$$y = \frac{1}{\mu(t)}\left(\int \mu(t)g(t)\,dt + C\right)$$

Step 7: If applicable, plug in initial conditions to solve for C.

While you can follow these steps to solve any first-order linear equation, make sure you're practicing problems and *understanding* the method versus just memorizing it.

> **Example:** Find the general solution to the differential equation.
>
> $$t^3y' + t^2y = t^3, \quad t > 0$$

Solution: Following the steps outlined above:

Step 1: Divide both sides by t^3 to get equation in the correct form.

$$t^3y' + t^2y = t^3 \quad \Longrightarrow \quad \frac{t^3y' + t^2y}{t^3} = \frac{t^3}{t^3} \quad \Longrightarrow \quad y' + \frac{1}{t}y = 1$$

Step 2: Calculate integrating factor.

$$\mu(t) = e^{\int p(t)\,dt} = e^{\int \frac{1}{t}\,dt} = e^{\ln(t)} = t$$

Step 3: Multiply integrating factor $\mu(t) = t$ to both sides.

$$y' + \frac{1}{t}y = 1 \quad \Longrightarrow \quad t\cdot\left(y' + \frac{1}{t}y\right) = t\cdot(1) \quad \Longrightarrow \quad ty' + y = t$$

Step 4: Use reverse product rule to simplify left-hand side.

$$ty' + y = t \quad \Longrightarrow \quad \frac{d}{dt}(ty) = t$$

Step 5: Integrate both sides.

$$\int \frac{d}{dt}(ty)\;dt = \int t\,dt \quad \Longrightarrow \quad ty = \frac{1}{2}t^2 + C$$

Step 6: Divide both sides by t to get general solution:

$$ty = \frac{1}{2}t^2 + C \quad \Longrightarrow \quad \frac{ty}{t} = \frac{\frac{1}{2}t^2 + C}{t} \quad \Longrightarrow \quad \boxed{y = \frac{1}{2}t + \frac{C}{t}}$$

Assigned Homework: 1-3, 9, 10, 13-15, 18, 20

Extra Practice: Find the general solution to each of the following differential equations.

1) $y' + 3t^2 y = 0$

2) $y' + 2y = t^3 e^{-2t}$

3) $ty' + (\ln t)y = 0$

2.2 Separable Equations

One of the nicer forms of differential equations are those that can be written as

$$-M(x) + N(y)\frac{dy}{dx} = 0.$$

After moving around terms and "multiplying" both sides by dx, we arrive at what we call a *separable* equation.

Separable Equation: $N(y)\,dy = M(x)\,dx$

Solving for these equations is very straightforward: simply integrate the left side with respect to y and the right side with respect to x. Typically, you will not be able to actually solve for y, but you will arrive at what is called an *implicit* solution for the differential equation. See the example below.

Example: Provide an implicit solution for the initial value problem

$$y' = -\frac{x}{y}, \quad y(0) = 1.$$

Solution: First, separate the variables.

$$y' = -\frac{x}{y} \quad \Longrightarrow \quad \frac{dy}{dx} = -\frac{x}{y} \quad \Longrightarrow \quad y\,dy = -x\,dx$$

We now see that our equation is separable, so we next integrate both sides.

$$y\,dy = -x\,dx \quad \Longrightarrow \quad \int y\,dy = \int -x\,dx \quad \Longrightarrow \quad \frac{y^2}{2} = -\frac{x^2}{2} + C$$

Finally, we solve for C by plugging in the initial condition.

$$y(1) = 0 \quad \Longrightarrow \quad \frac{0^2}{2} = -\frac{1^2}{2} + C \quad \Longrightarrow \quad C = \frac{1}{2}$$

Our implicit solution to the IVP is thus

$$\boxed{\frac{y^2}{2} = -\frac{x^2}{2} + \frac{1}{2}}$$

or equivalently,

$$\boxed{x^2 + y^2 = 1}$$

(Try to see why the two solutions are equivalent. Can you imagine what the resultant solution looks like?)

Assigned Homework: 1, 3, 4, 5, 6, 9-11, 13, 17

Extra Practice:

1) Find an implicit solution to the following IVP.

$$y' = \frac{2x+1}{5y^4+1}, \quad y(2) = 1$$

2) Find the general solution to the differential equation.

$$y' = 2xy^2$$

2.3 Modeling with First Order Equations

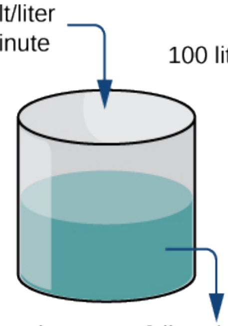
This class-favorite section is concerned with solving applied problems using the methods described above. While there are a myriad of ways one can apply first order differential equations to real-life problems, we will only be focusing on one type of application for this section— tank/mixing problems.

Suppose a homogeneous mixture containing a solvent flows into a tank at some rate. The mixture is well-stirred while inside the tank and is allowed to leave at some other rate. We are interested in modeling the amount of solvent, $Q(t)$, that is present in the tank at *any* time t. We can achieve this knowing the following fact about the *rate of change*, $\frac{dQ}{dt}$, of the solvent:

$$\boxed{\frac{dQ}{dt} = (\text{rate coming in}) - (\text{rate going out})}$$

The above equation leads to a first order differential equation, which we can usually solve. This is best understood with an example.

> **Example:** Suppose a 100 L tank initially contains 4 kg of salt. Water containing $\frac{1}{2}$ kg/L salt is poured in to the tank at a rate of 2 L/min and the mixture leaves the tank at the same rate. Calculate the amount of salt in the tank at any time t.



Solution: Let $Q(t)$ be the amount of salt in the tank at time t. Recall the formula:

$$\boxed{\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})\,.}$$

Hence, to solve this problem, we must first figure out the rate at which salt *comes in* to the tank and the rate at which salt *leaves* the tank.

Water flows in at a rate of 2 L/min and this same water contains $\frac{1}{2}$ kg/L salt. Hence, the rate at which salt enters the tank is

$$\text{Rate in} = \frac{2\text{L}}{1\text{min}} \times \frac{1\text{kg}}{2\text{L}} = \frac{\cancel{2\text{L}}}{1\text{min}} \times \frac{1\text{kg}}{\cancel{2\text{L}}} = 1\frac{\text{kg}}{\text{min}}.$$

Water is flowing out at the same rate of 2 L/min. However, how much salt is leaving with this water? This is where these problems can get a little tricky. We know that since water is flowing in and out of the tank at the same rate, there will always be 100 L of water in the tank. We also know that there is *exactly* $Q(t)$ kg of salt in the tank at any time t. Therefore, there is exactly $Q(t)$ kg of salt per 100 L of water in the tank. We can now calculate the rate at which salt leaves the tank:

$$\text{Rate out} = \frac{2\text{L}}{1\text{min}} \times \frac{Q(t)\text{kg}}{100\text{L}} = \frac{2\cancel{\text{L}}}{1\text{min}} \times \frac{Q(t)\text{kg}}{100\cancel{\text{L}}} = \frac{Q(t)}{50}\frac{\text{kg}}{\text{L}}.$$

Subtracting the rate in and the rate out yields

$$\frac{dQ}{dt} = 1\frac{\text{kg}}{\text{min}} - \frac{Q(t)}{50}\frac{\text{kg}}{\text{L}}.$$

Or, more succinctly,

$$Q' = 1 - \frac{1}{50}Q.$$

We can rewrite the above equation as

$$Q' + \frac{1}{50}Q = 1.$$

We now have a first order linear differential equation, so we can solve it using the method of integrating factor. The integrating factor in this case is

$$\mu(t) = e^{\int \frac{1}{50}\,dt} = e^{t/50}.$$

We multiply $\mu(t)$ to both sides and solve:

$$\begin{aligned} e^{t/50}Q' + \frac{1}{50}e^{t/50}Q &= e^{t/50} \\ \implies \frac{d}{dt}\left(e^{t/50}Q\right) &= e^{t/50} \\ \implies \quad e^{t/50}Q &= \int e^{t/50}\,dt \\ \implies \quad e^{t/50}Q &= 50e^{t/50} + C \\ \implies \quad Q(t) &= 50 + Ce^{-t/50}. \end{aligned}$$

To finish the problem, we need to plug in our initial condition to find C. We were given that the tank initially contains 4 kg of salt. Hence, $Q(0) = 4$ and so

$$4 = 50 + Ce^{-0/50} \implies C = -46.$$

Therefore, the amount of salt in the tank at any time t is

$$\boxed{Q(t) = 50 - 46e^{-t/50}}$$

Now let's try a harder one.

Example: A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Water that contains 1/4 g of salt per liter is poured into the tank at a rate of 4 L/min and the mixture is drained from the tank at a rate of 2 L/min. Find the amount of salt in the tank at the point of overflowing.

Solution: We proceed as before, using the same formula

$$\boxed{\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})\,.}$$

Just like before, we can find the rate of salt *coming in* to the tank using the fact that water containing 1/4 g/L salt is flowing in at a rate of 4 L/min:

$$\text{Rate in} = \frac{4\text{L}}{1\text{min}} \times \frac{1\text{g}}{4\text{L}} = \frac{\cancel{4\text{L}}}{1\text{min}} \times \frac{1\text{g}}{\cancel{4\text{L}}} = 1\frac{\text{g}}{\text{min}}.$$

The rate out will be a bit more difficult. We know water is leaving the tank at a rate of 2 L/min. However, how much salt is in this water? Like before, it will depend on the amount of salt in the tank $Q(t)$ and the amount of water in the tank. In this case, though, the amount of water in the tank *changes* with time. For now, let's just say that there are $W(t)$ liters of water in the tank at time t, so that

$$\text{Rate out} = \frac{2\text{L}}{1\text{min}} \times \frac{Q(t)\text{g}}{W(t)\text{L}} = \frac{2\cancel{\text{L}}}{1\text{min}} \times \frac{Q(t)\text{g}}{W(t)\cancel{\text{L}}} = \frac{2Q(t)}{W(t)}\frac{\text{g}}{\text{min}}.$$

To find $W(t)$, we will think about it logically. We know that at time $t = 0$ there are 200 liters of water in the tank. For every minute thereafter, 4 liters of water enter and 2 liters leave, yielding an additional 2 liters of water in the tank *per minute*. So after t minutes, there are an additional $2t$ liters in the tank on top of the initial 200 liters that were already present. Hence, we find $W(t) = 200 + 2t$. Plugging this in gives us

$$\text{Rate out} = \frac{2Q(t)}{W(t)}\frac{\text{g}}{\text{min}} = \frac{2Q(t)}{200+2t}\frac{\text{g}}{\text{min}} = \frac{Q(t)}{100+t}\frac{\text{g}}{\text{min}}.$$

Now subtracting rate in and rate out gives us a differential equation:

$$\frac{dQ}{dt} = 1\frac{\text{g}}{\text{min}} - \frac{Q(t)}{100+t}\frac{\text{g}}{\text{min}},$$

which we can equivalently express as

$$Q' + \frac{1}{100+t}Q = 1.$$

Like before, we want to solve this equation using the method of integrating factor. In this case, the integrating factor is

$$\mu(t) = e^{\displaystyle\int \frac{1}{100+t}\,dt} = e^{\ln(t+100)} = t + 100.$$

Now multiply both sides by $\mu(t)$ and solve the differential equation:

$$\begin{aligned}
(t+100)Q' + Q &= t + 100 \\
\implies \frac{d}{dt}\left[(t+100)Q\right] &= t + 100 \\
\implies (t+100)Q &= \int t + 100\,dt \\
\implies (t+100)Q &= \frac{1}{2}t^2 + 100t + C \\
\implies Q(t) &= \frac{\frac{1}{2}t^2 + 100t + C}{t+100}.
\end{aligned}$$

To find C, we plug in the initial condition

$$Q(0) = 10 \qquad \Longrightarrow \qquad 10 = \frac{\frac{1}{2}0^2 + 100(0) + C}{0 + 100} \qquad \Longrightarrow \qquad C = 1000,$$

giving us the amount of salt in the tank at time t,

$$Q(t) = \frac{\frac{1}{2}t^2 + 100t + 1000}{t + 100}.$$

To finish the problem, we must find out how much salt is in the tank at the point of overflowing. We know that the tank has a capacity of 500 liters and that at any time t, there are exactly $W(t) = 200 + 2t$ liters of water in the tank. To find the *time* at which the tank overflows, equate 500 with $W(t)$ and solve:

$$500 = 2t + 200 \qquad \Longrightarrow \qquad t = 150.$$

Finally, we plug in $t = 150$ for $Q(t)$:

$$Q(150) = \frac{\frac{1}{2}150^2 + 100(150) + 1000}{150 + 100} = \boxed{109 \text{ grams of salt.}}$$

Note 1: You do not have a calculator on the exam, so I don't care if you simplify the end result.

Note 2: You can actually calculate the formula for $W(t)$ by solving yet another differential equation. You're given a rate of change of water in the tank, 2 L/min, and an initial condition $W(0) = 200$. Try to see if you can formulate a differential equation that gives the same result that we came up with above.

Assigned Homework: 1, 2, 3, 4, 5

Extra Practice:

1) A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with 1/2 pounds of salt per gallon is added to the tank at a rate of 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time t.

2) A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with 1/4 pounds of salt per gallon is added to the tank at a rate of 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank at the point of overflowing.

Chapter 3: Second Order Linear Equations

because first-order *apparently* just wasn't enough

3.1 Homogeneous Equations with Constant Coefficients

The first few sections of this chapter are concerned with second-order homogeneous differential equations with constant coefficients (a real mouthful). More succinctly, these are equations of the form

$$ay'' + by' + cy = 0.$$

Such differential equations can be solved by looking for roots of their corresponding *characteristic equations.*

Characteristic Equation: $ar^2 + br + c = 0$

There are three different cases to the characteristic equation. Either we have distinct real roots, complex conjugate roots, or repeated real roots. In this section, we will consider the first case. If r_1 and r_2 are distinct real roots to the characteristic equation, then the general solution to $ay'' + by' + cy = 0$ is

Case 1 ($\mathbf{r_1 \neq r_2}$) : $\quad y = c_1e^{r_1t} + c_2e^{r_2t}$

Solving the characteristic equation is just a matter of solving a quadratic equation, which shouldn't be too bad.

Example: Solve the initial value problem:

$$y'' + 6y' + 5y = 0, \quad y(0) = 3,\ y'(0) = -1.$$

Solution: Write out the characteristic equation and solve:

$$r^2 + 6r + 5 = 0 \quad \Longrightarrow \quad (r+1)(r+5) = 0 \quad \Longrightarrow \quad r_1 = -1,\ r_2 = -5.$$

We can now calculate the general solution and its derivative:

$$y(t) = c_1e^{-t} + c_2e^{-5t}$$
$$y'(t) = -c_1e^{-t} - 5c_2e^{-5t}$$

Plugging in the initial conditions $y(0) = 3$ and $y'(0) = -1$ allows us to solve for the constants c_1 and c_2 with two equations:

$$3 = c_1 + c_2 \qquad -1 = -c_1 - 5c_2$$

Solving with either elimination or substitution yields $c_1 = 7/2$ and $c_2 = -1/2$, so our solution is

$$\boxed{y = \frac{7}{2}e^{-t} - \frac{1}{2}e^{-5t}.}$$

Assigned Homework: 1, 3, 7, 13-16

Extra Practice:

1) Find a general solution to the differential equation.

$$10y'' - 3y' - y = 0$$

2) Solve the initial value problem:

$$6y'' - y' - y = 0, \quad y(0) = 10, \; y'(0) = 0.$$

3.3 Complex Roots of the Characteristic Equation

If the characteristic equation $ar^2 + br + c = 0$ has *complex* roots $r = \alpha \pm \beta i$, then the general solution to $ay'' + by' + cy = 0$ is

Case 2 $(\mathbf{r} = \alpha \pm \beta\mathbf{i})$: $\quad y = c_1 e^{\alpha t}\cos(\beta t) + c_2 e^{\alpha t}\sin(\beta t)$

Solving these types of equations boils down to using the quadratic formula. Just in case you've forgotten, here it is:

Quadratic Formula: $r = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example: Find the general solution to the equation.

$$y'' + 4y' + 13y = 0$$

Solution: Just like before, we write out the characteristic equation and solve:

$$r^2 + 4r + 13 = 0 \quad \Longrightarrow \quad r = \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)} \quad \Longrightarrow \quad r = -2 \pm 3i.$$

Thus, the general solution:

$$\boxed{y = c_1 e^{-2t}\cos(3t) + c_2 e^{-2t}\sin(3t).}$$

Assigned Homework: 7, 10, 12, 18-22

Extra Practice:

1) Find the general solution the equation.

$$y'' - 2y' + 3y = 0$$

2) Solve the initial value problem:

$$y'' + 14y' + 50y = 0, \quad y(0) = 2, \; y'(0) = -17.$$

3.4 Repeated Roots

... roots ...

If the characteristic equation $ar^2 + br + c = 0$ has *repeated* roots, i.e. real roots $r_1 = r_2$, then the general solution to $ay'' + by' + cy = 0$ is

$$\textbf{Case 3 } (\mathbf{r_1 = r_2 = r}): \quad y = c_1 e^{rt} + c_2 t e^{rt}$$

We've now covered all three cases of the charactersitic equation.

Example: Solve the initial value problem:

$$y'' + 6y' + 9y = 0, \quad y(0) = 3, \; y'(0) = -1.$$

Solution: Yet again we solve the characteristic equation:

$$r^2 + 6r + 9 = 0 \quad \Longrightarrow \quad (r+3)^2 = 0 \quad \Longrightarrow \quad r_1 = r_2 = -3.$$

Hence, we can calculate the general solution and its derivative:

$$y(t) = c_1 e^{-3t} + c_2 t e^{-3t}$$
$$y'(t) = -3c_1 e^{-3t} + c_2(1 - 3t)e^{-3t}.$$

Plugging in the initial conditions $y(0) = 3$ and $y'(0) = -1$ allows us to solve for c_1 and c_2 through the two equations

$$\begin{aligned} &3 = c_1 + c_2(0) \\ \implies &3 = c_1 \end{aligned} \qquad\qquad \begin{aligned} &-1 = -3c_1 + c_2(1 - 3(0)) \\ \implies &c_2 = 8. \end{aligned}$$

Therefore, the solution to the IVP is

$$\boxed{y = 3e^{-3t} + 8te^{-3t}.}$$

Note: Technically Section 3.4 also covers reduction of order, but that won't be on the final exam since I've bullied you enough this semester.

Assigned Homework: 1, 2, 4, 6, 8, 9, 11, 12, 14

Extra Practice:

1) Find the general solution to the equation.

$$y'' - 4y' + 4y = 0$$

2) Solve the IVP:

$$4y'' - 12y' + 9y = 0, \quad y(0) = 3, \; y'(0) = 5/2.$$

3.5 Method of Undetermined Coefficients

It's now time to venture out of our comfort zone and attempt to tackle some non-homogeneous equations. Specifically, we're going to look at equations of the form

$$\boxed{ay'' + by' + cy = g(t)}$$

As it turns out, if $y_h = c_1y_1 + c_2y_2$ is the general solution to the homogeneous equation $ay'' + by' + cy = 0$, then a solution of the non-homogeneous equation can be expressed as

$$\boxed{y = c_1y_1 + c_2y_2 + y_p}$$

where y_p is called the *particular* solution. The method of undetermined coefficients is used to find this particular solution y_p. The following is a step-by-step method to solve an equation using the method of undetermined coefficients:

Step 1: Solve the homogeneous equation $ay'' + by' + cy = 0$:

$$y_h = c_1y_1 + c_2y_2$$

Step 2: "Guess" the form of y_p, i.e. express the particular solution in a form that matches the non-homogeneous term $g(t)$. (See table below)

Step 3: Check to see if your guess matches either part of the homogeneous solution y_1 or y_2. If so, multiply that particular guess by t. Repeat this step until no part of the guess matches with the homogeneous solution.

Step 4: Plug in your guess y_p into the original differential equation and solve.

The hardest part of this method is determining a suitable guess for the particular solution. Again, the idea is to formulate a guess around the form of $g(t)$. The following table may help a bit for the guessing part:

$\mathbf{g(t)}$	Initial Guess for $\mathbf{y_p}$
$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$	$A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n$
$(a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n)e^{\alpha t}$	$(A_0 + A_1 t + A_2 t^2 + \cdots + A_n t^n)e^{\alpha t}$
$(a_0 + a_1 t \cdots + a_n t^n)e^{\alpha t} \begin{cases}\cos(\beta t)\\ \sin(\beta t)\end{cases}$	$(A_0 + A_1 t + \cdots + A_n t^n)e^{\alpha t}\cos(\beta t)$ $+(B_0 + B_1 t + \cdots + B_n t^n)e^{\alpha t}\sin(\beta t)$

That table can look pretty daunting, so here are some concrete examples:

$\mathbf{g(t)}$	Initial Guess for $\mathbf{y_p}$
$5e^{3t}$	Ae^{3t}
$2t$	$A_0 + A_1 t$
$(18t + 1001t^2)\, e^{-t}$	$(A_0 + A_1 t + A_2 t^2)\, e^{-t}$
$t\sin(2t)$	$(A_0 + A_1 t)\cos(2t) + (B_0 + B_1 t)\sin(2t)$
6.152819571	A
$6.152819571\, t^3$	$A_0 + A_1 t + A_2 t^2 + A_3 t^3$
$e^{\sqrt{2}t}\cos(\sqrt{3}t)$	$Ae^{\sqrt{2}t}\cos(\sqrt{3}t) + Be^{\sqrt{2}t}\sin(\sqrt{3}t)$

Alright, let's try solving a problem.

Example: Find the general solution to the differential equation.

$$y'' - 7y' + 12y = 5e^{4t}.$$

Solution: Let's follow the steps outlined above:

Step 1: Solve the homogeneous equation $y'' - 7y' + 12y = 0$. Try this on your own; you should get

$$y_h = c_1 e^{3t} + c_2 e^{4t}.$$

Step 2: "Guess" the form of y_p. In this case, $g(t) = 5e^{4t}$, so our initial guess will be $y_p = Ae^{4t}$.

Step 3: Check to see if the guess Ae^{4t} matches with anything in the homogeneous solution $y_h = c_1 e^{3t} + c_2 e^{4t}$. In this case, Ae^{4t} matches with $c_2 e^{4t}$, so we have to *modify* our guess:

$$y_p = Ae^{4t} \xrightarrow{\text{modify}} y_p = Ate^{4t}$$

Step 3: We repeat Step 3 again. Now our guess is $y_p = Ate^{4t}$ and we want to see if anything matches with $y_h = c_1 e^{3t} + c_2 e^{4t}$. In this case, nothing matches so we can move on to the last step.

Step 4: We now plug in our guess $y_p = Ate^{4t}$ into the differential equation and solve for A. Let's first calculate the derivtaives of y_p:

$$\begin{aligned} y_p &= Ate^{4t} \\ y_p' &= (A + 4At)\, e^{4t} \\ y_p'' &= (8A + 16At)\, e^{4t}. \end{aligned}$$

Plug these in to the differential equation $y'' - 7y' + 12y = 5e^{4t}$ and solve for A:

$$\begin{aligned} (8A + 16At)\, e^{4t} - 7\,(A + 4At)\, e^{4t} + 12Ate^{4t} &= 5e^{4t} \\ \implies \frac{(8A + 16At)\, e^{4t} - 7\,(A + 4At)\, e^{4t} + 12Ate^{4t}}{e^{4t}} &= \frac{5e^{4t}}{e^{4t}} \\ \implies 8A + 16At - 7A - 28At + 12At &= 5 \\ \implies A &= 5. \end{aligned}$$

Therefore, $y_p = 5te^{4t}$ and so our general solution is

$$\boxed{y = c_1 e^{3t} + c_2 e^{4t} + 5te^{4t}.}$$

These are definitely a lot of work. Let's try one where we only set up the guess, but don't actually solve for the coefficients.

Example: For the following equation, determine a suitable form for the particular solution $y_p(t)$ if the method of undetermined coefficients is to be used.

$$y'' - 6y' + 9y = (1 + t^2)e^{3t}$$

Solution: Follow the steps!

Step 1: Solve the homogeneous equation $y'' - 6y' + 9y = 0$. We have repeated roots this time:

$$y_h = c_1 e^{3t} + c_2 t e^{3t}.$$

Step 2: "Guess" the form of y_p. In this case $g(t) = (1 + t^2)e^{3t}$, which is just a quadratic times e^{3t}. Hence, our initial guess will be $y_p = (A_0 + A_1 t + A_2 t^2)e^{3t}$.

Step 3: Check to see if the guess $(A_0 + A_1 t + A_2 t^2)e^{3t}$ matches with anything in the homogeneous solution $y_h = c_1 e^{3t} + c_2 t e^{3t}$. Right away. we see that $A_0 e^{3t}$ matches with $c_1 e^{3t}$ and so we have to modify our guess:

$$y_p = (A_0 + A_1 t + A_2 t^2)e^{3t} \xrightarrow{\text{modify}} y_p = t(A_0 + A_1 t + A_2 t^2)e^{3t}$$

Step 3: Repeat Step 3 again. Now our guess is $y_p = t(A_0 + A_1 t + A_2 t^2)e^{3t}$ and we want to see if anything matches with $y_h = c_1 e^{3t} + c_2 t e^{3t}$. Yet again, we have match. This time the term $A_0 t e^{3t}$ matches with $c_2 t e^{3t}$ and so we have to modify our guess *again*:

$$y_p = t(A_0 + A_1 t + A_2 t^2)e^{3t} \xrightarrow{\text{modify}} y_p = t^2(A_0 + A_1 t + A_2 t^2)e^{3t}$$

Step 3: Since we modified, we must repeat Step 3 again. Let's see if our new guess $y_p = t^2(A_0 + A_1 t + A_2 t^2)e^{3t}$ matches with $y_h = c_1 e^{3t} + c_2 t e^{3t}$. This time there are no matches! We stow away our magnifying glasses in triumphant victory, since we now have the suitable form for the particular solution:

$$\boxed{y_p = t^2(A_0 + A_1 t + A_2 t^2)e^{3t}}$$

Assigned Homework: 1-5, 7, 9, 15, 16, 21-23(a)

Extra Practice: For the following three differential equations, determine a suitable form for the particular solution if the method of undetermined coefficients is to be used. For Exercise 3), also find the general solution.

1) $y'' + y' - 2y = te^{3t}\cos(5t)$

2) $y'' + y = 3t^2 \sin(t)$

3) $y'' - 6y' + 8y = (11 - 6t)e^{t}$

Chapter 4: Higher Order Linear Equations

ok now this is just getting out of hand

4.2 Homogeneous Equations with Constant Coefficients

We only covered one section in Chapter 4, but it's a crucial one. In this section, we will be solving higher-order homogeneous differential equations with constant coefficients, i.e. equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0$$

Like in Chapter 2, we're going to be using the characteristic equation to solve these. Unlike Chapter 2, this equation is typically very difficult to solve.

Characteristic Equation: $a_n r^n + a_{n-1} r^{n-1} + \cdots + a_2 r^2 + a_1 r + a_0 = 0$

Roots of the characteristic equation will determine the general solution. The following table details the possible roots for the characteristic equation.

Roots of Characteristic Equation	Corresponding Solution
Distinct Real Roots: $r_1 \neq r_2 \neq \cdots \neq r_n$	$e^{r_1 t}, e^{r_2 t} \ldots, e^{r_n t}$
Distinct Complex Conjugate Pairs: $\alpha_1 \pm \beta_1 i \neq \cdots \neq \alpha_n \pm \beta_n i$	$e^{\alpha_1 t}\cos(\beta_1 t), e^{\alpha_1 t}\sin(\beta_1 t), \ldots,$ $e^{\alpha_n t}\cos(\beta_n t), e^{\alpha_n t}\sin(\beta_n t)$
Repeated Real Roots: $r = r_1 = r_2 = \cdots = r_n$	$e^{rt}, te^{rt}, \ldots, t^{n-1}e^{rt}$
Repeated Complex Conjugate Pairs: $\alpha \pm \beta i = \alpha_1 \pm \beta_1 i = \cdots = \alpha_n \pm \beta_n i$	$e^{\alpha t}\cos(\beta t), e^{\alpha t}\sin(\beta t),$ $te^{\alpha t}\cos(\beta t), te^{\alpha t}\sin(\beta t), \ldots,$ $t^{n-1}e^{\alpha t}\cos(\beta t), t^{n-1}e^{\alpha t}\sin(\beta t)$

Usually, we'll need to factor by grouping, perform a substitution, or use synthetic division to solve these kinds of problems.

Example: Find the general solution of the differential equation.

$$3y''' - 3y'' - y' + y = 0$$

Solution: Let's write out the characteristic equation.

$$3r^3 - 3r^2 - r + 1 = 0$$

In this case, we'll want to perform factoring by grouping:

$$3r^3 - 3r^2 - r + 1 = 0 \quad \Longrightarrow \quad 3r^2(r-1) - (r-1) = 0 \quad \Longrightarrow \quad (3r^2 - 1)(r-1) = 0$$

Solving for r gives us three solutions:

$$r_1 = \frac{\sqrt{3}}{3}, \quad r_2 = -\frac{\sqrt{3}}{3}, \quad r_3 = 1.$$

Thus, the general solution is

$$\boxed{y = c_1 e^{\sqrt{3}/3t} + c_2 e^{-\sqrt{3}/3t} + c_3 e^t.}$$

Example: Find the general solution of the differential equation.

$$y^{(4)} + 2y'' + y = 0$$

Solution: Like before, we need to set up the characteristic equation:

$$r^4 + 2r^2 + 1 = 0.$$

If we substitute $s = r^2$, then we get

$$r^4 + 2r^2 + 1 = 0 \quad \Longrightarrow \quad s^2 + 2s + 1 = 0 \quad \Longrightarrow \quad (s+1)^2 = 0.$$

Substituting r back into the equation allows us to solve:

$$(r^2 + 1)^2 = 0 \quad \Longrightarrow \quad r_1 = i,\ r_2 = -i,\ r_3 = i,\ r_4 = -i.$$

In this case, we have a complex conjugate pair $\pm i$ that repeats, so the general solution is

$$\boxed{y = c_1 \cos(t) + c_2 \sin(t) + c_3 t \cos(t) + c_4 t \sin(t).}$$

Example: Find the general solution of the differential equation.

$$y^{(4)} - 2y''' + y'' + 2y' - 2y = 0$$

Solution:
Like the previous two problems, we set up the characteristic equation.

$$r^4 - 2r^3 + r^2 + 2r - 2 = 0$$

Unfortunately, there's no clear way to factor this— neither grouping nor substitution will do the trick. We're going to have to rely on synthetic division in this case. To do this, we'll need to recall the rational roots theorem:

The Rational Roots Theorem

If r is a rational root of the polynomial $a_n x^n + \ldots a_1 x + a_0 = 0$, then

$$r = \frac{\text{Factor of } a_0}{\text{Factor of } a_n}.$$

In the case of our characteristic equation $r^4 - 2r^3 + r^2 + 2r - 2 = 0$, we have $a_0 = -2$ and $a_n = 1$. Let's list out the factors:

$$\begin{aligned} &\text{Factors of } a_0 = -2 : \pm 1, \pm 2 \\ &\text{Factors of } a_n = 1 \quad : \pm 1. \end{aligned}$$

Hence the possible roots are $\pm 1, \pm 2$. Let's try $r = 1$ and see if that's a root using synthetic division:

$$\begin{array}{c|ccccc} & 1 & -2 & 1 & 2 & -2 \\ 1 & & 1 & -1 & 0 & 2 \\ \hline & 1 & -1 & 0 & 2 & 0 \end{array}$$

Since we get 0 at the end, $r = 1$ is a root of the polynomial. This means that we can factor out the term $(r - 1)$ and we'll be left with the polynomial given by the coefficients on the bottom row, i.e.

$$r^4 - 2r^3 + r^2 + 2r - 2 = (r - 1)(r^3 - r^2 + 2).$$

Let's try synthetic division again to factor polynomial $r^3 - r^2 + 2$. The same possible roots $\pm 1, \pm 2$ apply in this case, coincidentally. Let's start with $r = 1$ again:

$$\begin{array}{c|cccc} & 1 & -1 & 0 & 2 \\ 1 & & 1 & 0 & 0 \\ \hline & 1 & 0 & 0 & 2 \end{array}$$

Since we didn't get a 0 in the last spot, $r = 1$ was not a root. So let's try $r = -1$:

$$\begin{array}{c|cccc} & 1 & -1 & 0 & 2 \\ -1 & & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & 0 \end{array}$$

That sweet, sweet 0 shows up in the last spot and so $r = -1$ is indeed a root. We can now factor out the term $(r + 1)$:

$$(r-1)(r^3 - r^2 + 2) = (r-1)(r+1)(r^2 - 2r + 2)$$

Since we're just left with a quadratic now, we can go ahead and solve for all of the possible roots using the quadratic formula:

$$r_1 = 1,\ r_2 = -1,\ r_3 = 1 + i,\ r_4 = 1 - i.$$

We finally have the general solution:

$$\boxed{y = c_1 e^t + c_2 e^{-t} + c_3 e^t \cos(t) + c_4 e^t \sin(t).}$$

Assigned Homework: 11, 13, 17, 19, 22, 23, 29, 32

Extra Practice: Find the general solution to the following differential equations.

1) $4y''' - 8y'' + 5y' - y = 0$

2) $16y^{(4)} - 72y'' + 81y = 0$

3) Solve the initial value problem:

$$y''' - 2y'' + 4y' - 8y = 0, \quad y(0) = 2,\ y'(0) = -2,\ y''(0) = 0.$$

Chapter 6: The Laplace Transform

6.2 Solution of Initial Value Problems

The Laplace transform is a way in which we can *transform* one function $f(t)$ into another function $F(s)$. This transform is nice because it allows us to express differential equations of a function $y(t)$ into equivalent *algebraic* equations of the function $Y(s)$. The Laplace Transform is defined as follows:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt.$$

Note: You will **not** need to know this definition for the exam. A table of useful Laplace transforms can be found on the last page of the packet and will be provided for you during the exam.

Example: Calculate the inverse Laplace of $F(s)$, where

$$F(s) = \frac{3s+8}{s^2+2s+5}.$$

Solution:
We first complete the square

$$F(s) = \frac{3s+8}{s^2+2s+1+4} = \frac{3s+8}{(s+1)^2+4}.$$

This looks similar to #9 and #10 in the table, where $a = -1$ and $b = 2$. From the table, we're either going to want $s - a$ or b in the numerator, so let's try to do that:

$$\begin{aligned}
\frac{3s+8}{(s+1)^2+4} &= \frac{3s}{(s+1)^2+4} + \frac{8}{(s+1)^2+4} \\
&= 3\frac{(s+1)-1}{(s+1)^2+4} + \frac{8}{(s+1)^2+4} \\
&= 3\frac{(s+1)}{(s+1)^2+4} - \frac{3}{(s+1)^2+4} + \frac{8}{(s+1)^2+4} \\
&= 3\frac{(s+1)}{(s+1)^2+4} + 5\frac{1}{(s+1)^2+4} \\
&= 3\frac{(s+1)}{(s+1)^2+4} + \frac{5}{2}\frac{2}{(s+1)^2+4}.
\end{aligned}$$

We can now take the inverse Laplace:

$$\begin{aligned} f(t) = \mathcal{L}^{-1}(F(s)) &= 3\mathcal{L}^{-1}\left(\frac{(s+1)}{(s+1)^2+4}\right) + \frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right) \\ &= \boxed{3e^{-t}\cos(2t) + \frac{5}{2}e^{-t}\sin(2t)}. \end{aligned}$$

The most important fact about the Laplace transform is its action on derivatives. If Y is the Laplace transform of a twice differentiable function y, then

$$\mathcal{L}\{y'\} = sY - y(0)$$

and

$$\mathcal{L}\{y''\} = s^2Y - sy(0) - y'(0).$$

The above two equations show how the derivative, an inherently calculus-based concept, gets turned into a polynomial, a purely *algebraic* concept.

Example: Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2,\ y'(0) = 3.$$

Solution: For every IVP with Laplace, we're going to want to take the Laplace transform of both sides and then solve for Y:

$$\begin{aligned} \mathcal{L}\{y'' - 6y' + 5y\} &= \mathcal{L}\left\{3e^{2t}\right\} \\ \mathcal{L}\{y''\} - 6\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} &= 3\mathcal{L}\left\{e^{2t}\right\} \\ \left(s^2Y - sy(0) - y'(0)\right) - 6(sY - y(0)) + 5Y &= \frac{3}{s-2} \\ s^2Y - 2s - 3 - 6sY + 12 + 5Y &= \frac{3}{s-2} \\ \left(s^2 - 6s + 5\right)Y - 2s + 9 &= \frac{3}{s-2} \\ \left(s^2 - 6s + 5\right)Y &= \frac{3}{s-2} + 2s - 9 \\ Y &= \frac{3}{(s-2)(s^2-6s+5)} + \frac{2s-9}{s^2-6s+5} \\ Y &= \frac{3}{(s-2)(s-5)(s-1)} + \frac{2s-9}{(s-5)(s-1)} \\ Y &= \frac{3}{(s-2)(s-5)(s-1)} + \frac{(2s-9)(s-2)}{(s-2)(s-5)(s-1)} \\ Y &= \frac{2s^2 - 13s + 21}{(s-2)(s-5)(s-1)}. \end{aligned}$$

We now use partial fractions on the last expression. Use whatever method you'd prefer, but you should end up with

$$\frac{2s^2 - 13s + 21}{(s-2)(s-5)(s-1)} = \frac{A}{s-2} + \frac{B}{s-5} + \frac{C}{s-1} = \frac{1}{s-2} + \frac{1/2}{s-5} + \frac{5/2}{s-1}.$$

Now we can solve:

$$Y = \frac{1}{s-2} + \frac{1/2}{s-5} + \frac{5/2}{s-1}$$

$$\implies \quad \mathcal{L}^{-1}(Y) = \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) + \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-5}\right) + \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right)$$

$$\boxed{y(t) = e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^{t}.}$$

Assigned Homework: 1-9(odds), 11-17, 19, 21, 23

Extra Practice: Solve the following IVPs using the Laplace transform.

1) $y'' + y = \sin(2t), \quad y(0) = 0, \; y'(0) = 1$

2) $2y'' + 2y' + y = 2y, \quad y(0) = 1, \; y'(0) = -1$

6.3 Step Functions

The *unit-step* function, *Heaviside* function, or *activation* function is defined as follows:

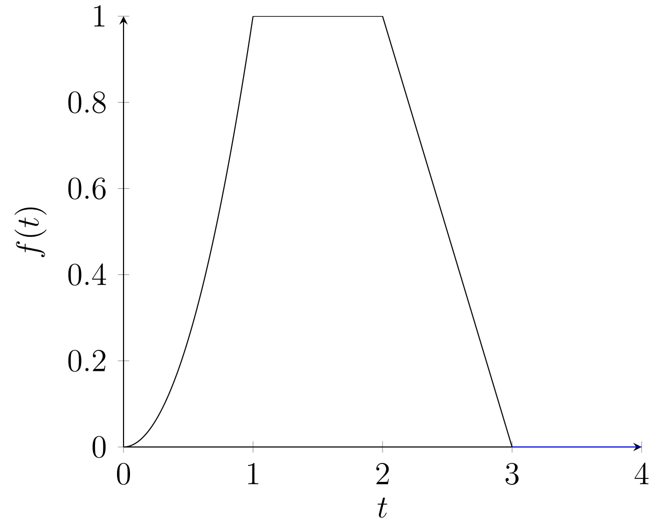
$$u_c(t) = \begin{cases} 0, & 0 \le t < c \\ 1, & t \ge c \end{cases}$$

The purpose of the step function is to be able to express piecewise functions in a more convenient way.

Example: Express $f(t)$ in terms of the unit-step function $u_c(t)$.

$$f(t) = \begin{cases} t^2, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ -(t+3), & t \ge 2. \end{cases}$$

Solution: While it's not required, let's graph this function for fun.



To express $f(t)$ in terms of step functions, we will use the *activate-deactivate* method. Notice that $f(t)$ can be decomposed into three pieces: one from $0 \leq t < 1$, one from $1 \leq t < 2$, and the last from $t \geq 2$. If we can express all three of these pieces individually, then $f(t)$ will be their sum. In other words, we have

$$\begin{aligned} f(t) &= t^2\left(u_0(t) - u_1(t)\right) + 1\left(u_1(t) - u_2(t)\right) - (t+3)u_2(t) \\ &= \boxed{t^2 + (1-t^2)u_1(t) - (t+4)u_2(t)}. \end{aligned}$$

Two transforms of particular importance in the Laplace table are the Laplace transform of the step function

$$\boxed{\mathcal{L}\left\{u_c(t)\right\} = \frac{1}{s}e^{-cs}}$$

as well as the Laplace transform of a step function attached to another function

$$\boxed{\mathcal{L}\left\{u_c(t)f(t-c)\right\} = F(s)\,e^{-cs}}$$

where $F(s) = \mathcal{L}\left\{f(t)\right\}$.

Example: Calculate the Laplace transform of the given function.

$$g(t) = \begin{cases} 2t+1, & 0 \leq t < 2 \\ 3t, & t \geq 2. \end{cases}$$

Solution: Start by expressing $g(t)$ in terms of step functions.

$$g(t) = (2t+1)\left(1 - u_2(t)\right) + 3tu_2(t) = (2t+1) + (t-1)u_2(t).$$

We can calculate the Laplace of the first term $(2t+1)$ using the table. For the second term, $(t-1)u_2(t)$ we will need the formula $\mathcal{L}\{u_c(t)f(t-c)\} = F(s)\,e^{-cs}$. In this case, we see that $c = 2$ and $f(t-c) = f(t-2) = t-1$. To find $f(t)$, we perform a substituion:

$$f(t-2) = t-2 \implies u = t-2$$
$$t = u+2 \implies f(u) = (u+2) - 2 = u.$$

Hence if $f(u) = u$, then $f(t) = t$. We know that $F(s) = \mathcal{L}\{t\} = 1/s^2$, and so we can finish the problem:

$$\mathcal{L}\left(g(t)\right) = \mathcal{L}\{2t+1\} + \mathcal{L}\{(t-1)u_2(t)\} = \boxed{\frac{2}{s^2} + \frac{1}{s} + \frac{1}{s^2}e^{-2s}}.$$

Example: Calculate the inverse Laplace transform of the following function.

$$F(s) = e^{-4s}\left(\frac{4}{s^3} + \frac{1}{s}\right)$$

We again use the formula from the table

$$\mathcal{L}^{-1}\left(e^{-cs}H(s)\right) = u_c(t)h(t-c).$$

In this case, we have $c = 4$ and $H(s) = 4/s^3 + 1/s$. We wish to find $h(t)$, the inverse Laplace of $H(s)$:

$$H(s) = \frac{4}{s^3} + \frac{1}{s} = 2\frac{2}{s^3} + \frac{1}{s} \qquad \implies \qquad h(t) = 2\mathcal{L}^{-1}\left(\frac{2}{s^3}\right) + \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 2t^2 + 1.$$

The formula, however, calls for $h(t-c)$ instead of $h(t)$. Since $c = 4$, we have

$$h(t-4) = 2\left(t-4\right)^2 + 1 = 2t^2 - 16t + 33,$$

and so

$$\mathcal{L}^{-1}\left(F(s)\right) = \boxed{u_4(t)\left(2t^2 - 16t + 33\right)}.$$

Assigned Homework: 1, 2, 8, 10, 12, 13-18, 19-24

Extra Practice:
1) Calculate the Laplace transform of the given function.

$$g(t) = \begin{cases} e^{-t}, & 0 \le t < 1 \\ e^{-2t}, & t \ge 1 \end{cases}$$

2) Calculate the inverse Laplace transform of the given function.

$$F(s) = e^{-2s}\left[\frac{3(s-3)}{(s+1)(s-2)} - \frac{s+1}{(s-1)(s-2)}\right]$$

6.4 Differential Equations with Discontinuous Forcing Functions

The idea of this section is to combine the previous two sections together, i.e. solving IVPs with step functions in them. This is the finale of the course and as such requires the most tedious computations. We'll look at one example here.

> **Example:** Solve the IVP:
>
> $$y'' - y = f(t), \quad y(0) = -1,\ y'(0) = 2,$$
>
> where
>
> $$f(t) = \begin{cases} t, & 0 \le t < 1 \\ 1, & t \ge 1. \end{cases}$$

Solution:
Right away we see that we'll need to convert $f(t)$ into step functions:

$$f(t) = t\,(1 - u_1(t)) + 1\,(u_1(t)) = t + (1-t)u_1(t).$$

Now we just take the Laplace of both sides and solve for Y:

$$\begin{aligned} y'' - y &= f(t) \\ y'' - y &= t + (1-t)u_1(t) \\ \mathcal{L}\{y''\} - \mathcal{L}\{y\} &= \mathcal{L}\{t\} + \mathcal{L}\{(1-t)u_1(t)\} \\ \left(s^2Y - sy(0) - y'(0)\right) - Y &= \frac{1}{s^2} - \frac{1}{s^2}e^{-s} \\ (s^2-1)Y + s - 2 &= \frac{1}{s^2} - \frac{1}{s^2}e^{-s} \\ Y &= \frac{-s+2}{s^2-1} + \frac{1}{s^2(s^2-1)} - \frac{1}{s^2(s^2-1)}e^{-s} \\ Y &= \frac{-s+2}{(s-1)(s+1)} + \frac{1}{s^2(s-1)(s+1)} - \frac{1}{s^2(s-1)(s+1)}e^{-s}. \end{aligned}$$

We use partial fractions work with the right-hand side:

$$\frac{-s+2}{(s-1)(s+1)} = \frac{A}{s-1} + \frac{B}{s+1} = \frac{1}{2}\frac{1}{s-1} - \frac{3}{2}\frac{1}{s+1}$$

and

$$\frac{1}{s^2(s-1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-1} + \frac{D}{s+1} = -\frac{1}{s^2} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} = H(s)$$

and so

$$Y = \frac{1}{2}\frac{1}{s-1} - \frac{3}{2}\frac{1}{s+1} + H(s) - H(s)\,e^{-s}.$$

Now recall the necessary formula,

$$\mathcal{L}^{-1}\left(H(S)e^{-cs}\right) = u_c(t)h(t-c).$$

In this case, we see that $c = 1$. We need to first find $h(t)$, which is the inverse Laplace transform of $H(s)$:

$$h(t) = \mathcal{L}^{-1}\left(H(s)\right) = \mathcal{L}^{-1}\left(-\frac{1}{s^2} + \frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1}\right) = -t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}.$$

Now we can find $h(t-c)$:

$$h(t-c) = h(t-1) = -(t-1) + \frac{1}{2}e^{(t-1)} - \frac{1}{2}e^{-(t-1)}.$$

Putting everything together, we get

$$\begin{aligned}
Y &= \frac{1}{2}\frac{1}{s-1} - \frac{3}{2}\frac{1}{s+1} + H(s) - H(s)\,e^{-s} \\
\Longrightarrow y(t) &= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - \frac{3}{2}\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(H(s)\right) - \mathcal{L}^{-1}\left(H(s)e^{-s}\right) \\
y(t) &= \frac{1}{2}e^{t} - \frac{3}{2}e^{-t} + h(t) - u_1(t)h(t-1) \\
y(t) &= \frac{1}{2}e^{t} - \frac{3}{2}e^{-t} - t + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} + u_1(t)\left(-(t-1) + \frac{1}{2}e^{(t-1)} - \frac{1}{2}e^{-(t-1)}\right) \\
&\boxed{y(t) = e^{t} - 2e^{-t} - t + u_1(t)\left(-(t-1) + \frac{1}{2}e^{(t-1)} - \frac{1}{2}e^{-(t-1)}\right)}.
\end{aligned}$$

Assigned Homework: 1-10

Extra Practice: Solve the following initial value problems.

1) $y'' + y = f(t), \quad y(0) = 1,\ y'(0) = 0$, where

$$f(t) = \begin{cases} 3, & 0 \le t < 4 \\ 2t-5, & t \ge 4 \end{cases}$$

2) $y'' + 2y' + y = f(t), \quad y(0) = 3,\ y'(0) = -1$, where

$$f(t) = \begin{cases} e^{t}, & 0 \le t < 1 \\ e^{t}-1, & t \ge 1 \end{cases}$$

Solutions to Exercises:

2.1: 1) $y = Ce^{-t^3}$ 2) $y = e^{-2t}\left(\frac{1}{4}t^4 + C\right)$ 3) $y = Ce^{-\frac{1}{2}(\ln t)^2}$

2.2: 1) $y^5 + y = x^2 + x - 4$ 2) $y = -\dfrac{1}{x^2 + C}$

2.3: 1) $Q(t) = 30 - 20\, e^{-t/10}$ 2) 47.5 lbs

3.1: 1) $y = c_1 e^{-t/5} + c_2 e^{t/2}$ 2) $y = 4e^{t/2} + 6e^{-t/3}$

3.3: 1) $y = c_1 e^{t}\cos(\sqrt{2}t) + c_2 e^{t}\sin(\sqrt{2}t)$ 2) $y = 2e^{-7t}\cos(t) - 3e^{-7t}\sin(t)$

3.4: 1) $y = c_1 e^{2t} + c_2 t e^{2t}$ 2) $y = 3e^{3t/2} - 2te^{3t/2}$

3.5:

1) $y_p = (A_0 + A_1 t)\, e^{3t}\cos(5t) + (B_0 + B_1 t) e^{3t}\sin(5t)$

2) $y_p = t(A_0 + A_1 t + A_2 t^2)\cos(t) + t(B_0 + B_1 t + B_2 t^2)\sin(t)$

3) $y = c_1 e^{2t} + c_2 e^{4t} + (1 - 2t)e^t$

4.2: 1) $y = c_1 e^t + c_2 e^{1/2t} + c_3 t e^{1/2t}$ 2) $y = c_1 e^{3/2t} + c_2 t e^{3/2t} + c_3 e^{-3/2t} + c_4 t e^{-3/2t}$

3) $y = \cos(2t) - 2\sin(2t) + e^{2t}$

6.2: 1) $y = \frac{5}{3}\sin(t) - \frac{1}{3}\sin(2t)$ 2) $y = 5e^{-t/2}\cos(t/2) - e^{-t/2}\sin(t/2) + 2t - 4$

6.3: 1) $\dfrac{1 - e^{-(s+1)}}{s+1} + \dfrac{e^{-(s+2)}}{s+2}$ 2) $u_2(t)\left(4e^{-(t-2)} - 4e^{-2(t-2)} + 2e^{(t-2)}\right)$

6.4:

1) $y = 3 - 2\cos t + 2u_4(t)\,(t - 4 - \sin(t - 4))$

2) $y = \dfrac{1}{4}\left(e^t + e^{-t}(11 + 6t)\right) + u_1(t)\left(te^{-(t-1)} - 1\right)$

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$	Notes
1. 1	$\dfrac{1}{s}, \quad s > 0$	Sec. 6.1; Ex. 4
2. e^{at}	$\dfrac{1}{s-a}, \quad s > a$	Sec. 6.1; Ex. 5
3. t^n, $n =$ positive integer	$\dfrac{n!}{s^{n+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
4. t^p, $p > -1$	$\dfrac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$	Sec. 6.1; Prob. 31
5. $\sin at$	$\dfrac{a}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Ex. 7
6. $\cos at$	$\dfrac{s}{s^2+a^2}, \quad s > 0$	Sec. 6.1; Prob. 6
7. $\sinh at$	$\dfrac{a}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 8
8. $\cosh at$	$\dfrac{s}{s^2-a^2}, \quad s > \|a\|$	Sec. 6.1; Prob. 7
9. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 13
10. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}, \quad s > a$	Sec. 6.1; Prob. 14
11. $t^n e^{at}$, $n =$ positive integer	$\dfrac{n!}{(s-a)^{n+1}}, \quad s > a$	Sec. 6.1; Prob. 18
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}, \quad s > 0$	Sec. 6.3
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$	Sec. 6.3
14. $e^{ct}f(t)$	$F(s-c)$	Sec. 6.3
15. $f(ct)$	$\dfrac{1}{c}F\left(\dfrac{s}{c}\right), \quad c > 0$	Sec. 6.3; Prob. 25
16. $\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	Sec. 6.6
17. $\delta(t-c)$	e^{-cs}	Sec. 6.5
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$	Sec. 6.2; Cor. 6.2.2

Note: Explanations and notation, as well as assigned homework problems, were based on the in-class textbook [1]. Extra in-text problems were mostly taken from [2].

References

[1] Boyce, William E. and DiPrima, Richard C., *Elementary Differential Equations* (2012). John Wiley & Sons, 10th ed.

[2] Trench, William F., *Elementary Differential Equations with Boundary Value Problems* (2013). Faculty Authored and Edited Books & CDs. 9.