MAS 2103-001

Name: Practice Final Exam

Please be sure to read each question carefully and show ALL of your work.

Sections that may be covered on the Final Exam: 1.1 - 1.5, 1.7, 2.1 - 2.3, 3.1, 3.2, 4.2, 4.3, 4.5, 5.1 - 5.3, 6.1

Sections that will NOT be covered on the Final Exam: 1.8, 1.9, 4.1, 6.2 - 6.4

Problem 1: Solve the following system of linear equations using an augmented matrix.

$$\begin{aligned} x_1 - 2x_2 + 5x_3 &= 1 \\ 4x_1 - 5x_2 + 8x_3 &= -2 \\ -3x_1 + 3x_2 - 2x_3 &= 3 \end{aligned}$$

Problem 2: Consider the following matrix A.

$$A = \begin{bmatrix} 1 & -4 & 7 \\ 0 & 1 & -4 \\ 2 & -6 & 8 \end{bmatrix}$$

a) Find A^{-1} by using elementary row operations on the augmented matrix $[A|I]$

b) Solve the equation $Ax = b$ by using A^{-1} found in part(a), where

$$b = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Problem 3: Use elementary row operations to calculate the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 3 & 5 & -2 & 4 \\ -2 & -3 & 2 & 3 \\ 1 & 5 & -6 & 2 \end{bmatrix}$$

Problem 4: Consider the following matrix A.

$$A = \begin{bmatrix} 1 & 1 & -1 & -4 & 1 \\ 0 & 1 & -1 & -3 & 3 \\ 1 & -2 & 6 & 5 & 0 \\ 0 & -1 & 2 & 3 & -1 \end{bmatrix}$$

a) Find a basis for $\mathrm{Nul}(A)$ and calculate the nullity of A.

b) Find a basis for $\mathrm{Col}(A)$ and calculate the rank of A.

Problem 5: Consider the following subspace of $\mathbb{R}^4$

$$W = \left\{ \begin{bmatrix} 3a + 6b - c \\ 6a - 2b - 2c \\ -9a + 5b + 3c \\ -3a + b + c \end{bmatrix} : a, b, \text{and } c \text{ are real numbers.} \right\}$$

a) Find a set of vectors that spans W.

b) Find a basis for W.

c) Calculate the dimension of W.

Problem 6: Consider the following matrix A.

$$A = \begin{bmatrix} 2 & 7 \\ 7 & 2 \end{bmatrix}$$

a) Calculate the eigenvalues of A.

b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

Problem 7: List the eigenvalues with their multiplicity of the following matrix.

$$A = \begin{bmatrix} 4 & -7 & 1 & 2 \\ 0 & 3 & -4 & 6 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Problem 8: Diagonalize the following matrix A, if possible.

$$A = \begin{bmatrix} 5 & -2 & -2 \\ 1 & 2 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 9: Consider the following vectors $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} = \begin{bmatrix} 12 \\ 3 \\ -4 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

a) Calculate $\mathbf{u} \cdot \mathbf{v}$ and $||\mathbf{u}||$

b) Find a unit vector in the same direction as $\mathbf{v}$.

Solutions:

1. $\begin{bmatrix} -3 \\ -2 \\ 0 \end{bmatrix}$

2. a) $A^{-1} = \begin{bmatrix} -8 & -5 & 9/2 \\ -4 & -3 & 2 \\ -1 & -1 & 1/2 \end{bmatrix}$ b) $\begin{bmatrix} 5/2 \\ 1 \\ 1/2 \end{bmatrix}$

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4. a) Possible basis: $\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$, nullity = 2.

   b) Possible basis: $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 6 \\ 2 \end{bmatrix} \right\}$, rank = 3.

5. a) $W = \text{span} \left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 3 \\ 1 \end{bmatrix} \right\}$

   b) Possible basis for W : $\left\{ \begin{bmatrix} 3 \\ 6 \\ -9 \\ -3 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ 5 \\ 1 \end{bmatrix} \right\}$

   c) $\dim(W) = 2$

6. a) $\lambda_1 = -5$, $\lambda_2 = 9$

   b) Possible basis for $\mathbb{E}_{-5}$: $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$. Possible basis for $\mathbb{E}_9$: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$.

7. $\lambda_1 = 4$ (multiplicity = 1), $\lambda_2 = 3$ (multiplicity = 2), $\lambda_3 = -2$ (multiplicity = 1)

8. $A = PDP^{-1}$ where $P = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

9. a) $\mathbf{u} \cdot \mathbf{v} = 3, \quad ||u|| = 13$

   b) $\begin{bmatrix} \sqrt{22}/11 \\ -3\sqrt{22}/22 \\ 3\sqrt{22}/22 \end{bmatrix}$