Periodic Orbits of State-Dependent Delay Differential Equations

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In this work we present a method to find periodic orbits for state-dependent delay differential equations. This method is based on a Newton-Kantorovich algorithm and is illustrated in the case of the one-dimensional Cubic Ikeda Map. Though this work is mainly numerical, the techniques developed are aimed to be a-posteriori used in the frame of computer-assisted proof.

Keywords: Delay differential equation, state-dependent delay

1. Introduction

In dynamical systems, finding the locus of periodic orbits can be quite challenging. If we restrict our discussion to hyperbolic periodic orbits, the difficulty comes partially from the fact that the orbit may not be stable in the Lyapunov sense. Classical approaches often consist in relating periodic orbits to fixed points of Poincaré return maps or special solutions of initial value problems.

The context of the present work is that of state-dependent delay equations. More precisely, we consider the system

$$\begin{cases} \dot{y} &= F[y(t-\delta(y(t)))],\, t>0 \\ y(t) = y_0(t), & t \le 0, \end{cases}$$

where $F:\mathbb{R}^n \to \mathbb{R}^n$ is a smooth function and $y = y(t) \in \mathbb{R}^n$ represents the state variable. Moreover, y_0 is a smooth function that represents the initial condition and $\delta(y(t))$ the state-dependent delay. For the remainder of this work, we will restrict our study to the case $n = 1$, though the techniques developed in this article may be adapted to work in any dimension. While the context of this paper concerns any smooth function F, we will restrict our scope to the case where F is a polynomial function. As will be seen in the next section, this assumption simplifies our approach, and solutions are proposed to overcome the difficulties encountered when F is not a polynomial.

The (state-dependent) delay in this article is assumed to satisfy

$$\delta(y(t)) = \tau(1 - \varepsilon y(t)), \tag{1}$$

where $\tau > 0$ is called the unperturbed delay and ε is a small parameter. We will have a special interest in the so-called *state-dependent Cubic Ikeda system*:

$$\begin{cases} \dot{y} &= y(t-\delta(y)) - y^3(t-\delta(y)),\, t>0 \\ y(t) = y_0(t), & t \leq 0, \end{cases} \tag{2}$$

where $y(t) \in \mathbb{R}$. After a time rescaling, the system above is equivalent to

$$\begin{cases} \dot{x} &= \tau F(x(t-\hat{\delta}(x(t)))),\, t \geq 0 \\ x(t) = x_0(t), & t \leq 0, \end{cases} \tag{3}$$

where

$$x(t) = y(\tau t),\;\; x_0(t) = y_0(\tau t), \;\text{ and }\; \hat{\delta}(x) = 1 - \varepsilon x(t).$$

This means that τ is now seen as a parameter, and the unperturbed delay is rescaled to 1. We will later on remove the hat on δ when calling equation (3). Observe that the dependence of Eq. (3) upon the instantaneous variable $x(t)$ is (only) located in the delay and this dependence is moderated by the parameter ε, which is assumed to be small in magnitude. This assumption is essential in the context of our study. At the end of this article, we propose ways to waive this assumption and how we intend to study more general cases.

The purpose of this article is to present a technique to find periodic orbits for the system (3) above for several values of the parameters τ and ε with a particular interest when the periodic orbits are not Lyapunov stable and/or when the dynamics display some complexity. One of our interests is to construct a 'bridge' between delay differential equations with constant delay and state-dependent delay differential equations. For instance, the method proposed in this article will enable us to contruct a continuation from a periodic orbit of the system (3) when $\varepsilon = 0$ to a periodic orbit of the same system when $\varepsilon \neq 0$. However, for this specific study case (2) introduced above, we will also be able to find periodic orbits when $\varepsilon \neq 0$ that do not seem to admit any continuation toward $\varepsilon = 0$. It should also be noted that the goal of this work is not to find *all* periodic orbits for a specific system. This latter problem may be highly nontrivial, especially when the system is chaotic.

1.1. *Some history*

Delay Differential Equations are a topic of much interest in applied mathematics and engineering and have also become important in biology, ecology, and medicine. This interest is partly driven by the fact that they can provide realistic models for dynamical systems where there is a communication lag between subsystems. Since the year 2020 and the ensuing explosion of COVID-19 cases globally, the interest for this topic has increased drastically, as epidemic models are considered with delays when there is a substantially large gap between infection time and the first symptoms. We refer to the work on population models [Arino *et al.*, 1998; Blythe *et al.*, 1982; Dunkel, 1968; Freedman & Wu, 1992; Gopalsamy *et al.*, 1990; Kuang, 1993; Li & Kuang, 2001; MacDonald, 1989] and those on the amplitude of an oscillator [Furumochi, 1977; Hale, 1979; Herz, 1995; Norkin, 1972]. Each equation may have a single delay, multiple delays, constant delays or non-constant delays. This last situation occurs when the communication lag is determined by the state of the system and may be modeled by state-dependent delay differential equations. When dealing with Ordinary Differential Equations (ODEs, for short) on a finite-dimensional space, the feedback is instantaneous, and so the dimension of the system is also finite. However, when dealing with Delay Differential Equations (DDEs, for short), an initial history (sometimes called *history segment* or *past*) needs to be specified on an immediate past time interval to uniquely determine the evolution of the system, and therefore the space of initial conditions is infinite-dimensional. As a consequence, even in the simplest case of a scalar DDE, the techniques used to study the corresponding solutions are very different from those developed for ODEs, and solving a DDE is in general a more delicate task. When there is a finite number of constant delays, a DDE can be reformulated as an ODE on a Banach space and, while the problem is infinite-dimensional, many classical dynamical systems techniques can be used directly [Bánhelyi, 2007; Hale & Lunel, 1993; Hale, 1977]. However, the situation becomes more complicated when the delay itself depends on the current state of the system. For example, the state space for such a system is a manifold instead of a Banach space [Arino *et al.*, 1998; Hartung *et al.*, 2006; Lani-Wayda & Walther, 2016; Qesmi & Walther, 2009; Walther, 2008]. Recently, a number of authors have moved the theory of state-dependent DDEs forward using a-posteriori techniques from functional analysis, see for example [Hartung *et al.*, 2006] or [Gimeno *et al.*, 2023; Yang *et al.*, 2022; Gimeno *et al.*, 2021; Yang *et al.*, 2021; He & de la Llave, 2016, 2017] in the context of quasi-periodic and periodic solutions of a state-dependent DDEs with quasi-periodic forcing.

There is already a large set of references about the computation of orbits and, particularly, periodic orbits for DDEs. In total, these methods possess their own strengths and weaknesses, see for instance [Engelborghs *et al.*, 2002; Khasi *et al.*, 2014; Lenz *et al.*, 2014] and references therein. Some of these methods are theoretical while others are more technical. For instance, approaches from finite difference methods, (that, vaguely speaking, resemble Runge-Kutta's method) are proposed in [Bellen & Zennaro, 2003; Bogacki & Shampine, 1989]. Other approaches have been followed in various contexts, as for instance in [Groothedde & Mireles-James, 2017; Krauskopf *et al.*, 2005; Sedaghat *et al.*, 2012; Torelli, 1989], and some techniques work when the equation admits several delays [Kennedy, 2009; Qesmi & Walther, 2009]. To our best knowledge, the techniques mentioned do not seem to be appropriate to establish rigorous mathematical (computer-assisted) proofs. In the context of state-dependent DDEs, it is crucial to develop some numerical method with a-posteriori theory. It should be noted that the works in [Szczelina & Zgliczyński, 2018; van den Berg *et al.*, 2022] propose algorithms to rigorously solve Eq. (3) when the delay is constant. However, it is not clear how each algorithm could be adapted or modified in the context of state-dependent DDEs. This is where the gap between DDEs with constant delay and state-dependent DDEs stands: several methods work well in the former case, but not in the latter. Finally, it should be noted that in [Church, 2022], the author proposes an implicit method of steps with state-dependent delays and presents validated numerics to rigorously enclose solutions of initial-value problems. However, it is not clear how this approach could be used when searching for periodic orbits.

1.2. *A strategy*

Recall that a solution of (3) satisfies

$$\begin{cases} \dot{x} &= \tau F(x(t-\delta(x)), \, t > 0, \\ x(t) = x_0(t), & t \leq 0. \end{cases} \tag{4}$$

A classical approach when solving DDEs consists of using the method of steps, i.e., to compute the solution interval by interval. More precisely, assume that $x(t)$ is the solution of (4) on the interval $[0, \alpha]$ for some $\alpha > 0$. The search of the solution x of (4) for $t > \alpha$ amounts to solving

$$\begin{cases} \dot{z} = \tau F(z(t-\delta(z)), \quad t > 0, \\ z(t) = x_0(t+\alpha), \quad t \leq -\alpha, \quad \& \quad z(t) \; = \; x(t+\alpha), \quad -\alpha \leq t \leq 0, \end{cases}$$

writing $x(t) = z(t-\alpha)$ for $t > \alpha$. Our search will be for periodic orbits with period T bigger than the unperturbed delay, this latter being rescaled to 1 in our case. If T happens to be less than 1 (which has not been observed for the study case we consider in this present work), we then look for periodic orbits of period kT, where k is a positive integer and $kT > 1$. We will be more specific in the following remarks. Assume therefore that x is a periodic orbit of (3) with period $T = m\alpha$ with $1 < \alpha < 2$ and m a positive integer. We estimate $x = x(t)$ on each interval $[(j-1)\alpha, j\alpha]$, $j = 0, \dots, m-1$, with polynomial functions

$$y_0, \; y_1, \dots, y_{m-1}$$

of degree q defined on the interval $[-\alpha, 0]$ in the following manner: for each integer $j = 0, \dots, m-1$, we write

$$x(t) \sim y_j(t - j\alpha), \quad (j-1)\alpha \leq t \leq j\alpha.$$

The estimation "$\sim$" of the solution $x(t)$ on each interval is obtained thanks to a Chebyshev-Lagrange interpolation (see next section), which allows us to work in a finite-dimensional space. The accuracy of our computation is then checked a-posteriori (see section 3). Note that the resulting orbit is T-periodic if $x(t+T) \equiv x(t)$, which amounts to writing $y_0 = y_m$.

Since our search is for periodic orbits, it is reasonable to assume that each solution exists for all t and is bounded. We then choose a large real number $M > 1$ and search for solutions $x = x(t)$ such that

$$\begin{aligned}
&\text{(i)}: \ |x(t)| \leq M \quad \text{for all } t \geq -\alpha. \quad \text{We further assume}\\
&\text{(ii)}: \ |\varepsilon| \leq \varepsilon_0 = \frac{1}{10M}, \\
&\text{(iii)}: \ 11/10 \leq \alpha \leq 9/5, \qquad \text{i.e.,} \qquad 1+\varepsilon_0 M \leq \alpha \leq 2(1-\varepsilon_0 M).
\end{aligned} \tag{5}$$

In Eq. (4), we replace the function F by F_M where

$$\begin{cases} F_M(u) = F(u), \text{ if } \ |u| \leq M + \dfrac{1}{2}, \ \text{ and } \ \forall u \in \mathbb{R}, \\ |F_M(u)| \leq \displaystyle\sup_{|u|\leq M+1} |F(u)| = K_0, \qquad |F'_M(u)| \leq \sup_{|u|\leq M+1} |F'(u)| = K_1, \end{cases} \tag{6}$$

and where F_M is smooth. If $x = x(t)$ satisfies (5) and

$$\begin{cases} \dot{x} \quad = \tau F_M(x(t-1+\varepsilon x(t)), t > 0 \\ x(t) = x_0(t), \qquad\qquad\qquad t \leq 0, \end{cases} \tag{7}$$

then x also satisfies (4).

Remarks: The advantage of replacing F by F_M is that the right hand side of (7) is always bounded. However, at each time a solution of (7) is computed on an interval, we need to verify that this solution satisfies (5)(i). If it does not, we need to redo the whole set of computations with a larger value of M as many time as needed until (5)(i) is satisfied. Clearly, if such an M cannot be found, the solution of (4) is unbounded. Observe that this latter scenario does indeed occur for the study case under consideration in this article when the parameter τ is large (see discussion in Section 3).

For some values of the period T, the choice of α may not be evident, i.e., as mentioned above, when the period is less than 1 (or too close to 1). In this case, our search will be for periodic orbits of period kT where $kT > 1.10$. Indeed both k and m are to be chosen in such a way that

$$11/10 \leq kT/m = \alpha \leq 9/5.$$

It should be noted, however, that based upon the results obtained in many study cases, including the Cubic Ikeda Map (see discussion in the last section) and the Mackey-Glass equation [van den Berg *et al.*, 2022], the period T is never less than 1.

1.3. *Outline*

This article is organized as follows. In section 2, we introduce the Picard Operator, defined on a set of Lipschitz functions, and show that, for sufficiently small values of the parameter ε, this operator is a contraction. For each initial condition, the solution of the equation under consideration is therefore the unique fixed point of this operator. To make the computation of the solution feasible, we need to introduce the *Reduced* Picard Operator, a Lagrange-Chebyshev-interpolated version of the Picard Operator. Under suitable conditions on the number of nodes and the value of the parameter ε, we show that this new operator is also a contraction. Hence, the unique fixed point of this latter operator approximates the solution we are looking for. In section 3, we construct a Newton-like operator where periodic orbits of the system under consideration are in one-to-one correspondence with zeros of this operator. We end this article by illustrating our method for the Cubic Ikeda map and we display several Lissajous phase portraits for different values of τ and ε. Concluding remarks and a discussion of future work are added in the final section.

2. Two Picard Operators

The solution $x = x(t)$ of (7) satisfies

$$x(t) = x_0(0) + \tau \int_0^t F_M[x(s - 1 + \varepsilon x(s))]ds, \quad t > 0 \tag{8}$$

and

$$x(t) \equiv x_0(t), \quad -\alpha \leq t \leq 0.$$

Observe that if x_0 is a $\mathbf{C}^r$ initial condition (for $r \geq 0$), the solution of (8) is $\mathbf{C}^{r+1}$. Since our search is for periodic orbits, it is then reasonable to assume that x_0 is $\mathbf{C}^1$. Also observe from (5) that, independently from the choice of the initial condition x_0, the solution of (8) satisfies

$$|\dot{x}(t)| \leq b := \tau K_0 + 1, \tag{9}$$

for $t > 0$.

Let I be an interval. We say that $x \in \mathbf{C}^{1,b}(\mathrm{I})$ if x is continuous on I and differentiable on its interior with $|\dot{x}(t)| \leq b$. Let $\alpha > 1$ and b be chosen as in (9). For a given initial condition $x_0 \in \mathbf{C}^{1,b}([-\alpha, 0])$, we define

$$\mathbf{C}^{1,b}_{x_0}([0,\alpha]) = \{\mathbf{f} \in \mathbf{C}^{1,b}([0,\alpha]) \mid \mathbf{f}(0) = x_0(0)\}$$

and the following Picard Operator

$$\Psi_{x_0} : \mathbf{C}^{1,b}_{x_0}([0,\alpha]) \to \mathbf{C}^{1,b}_{x_0}([0,\alpha]), \ x \mapsto \Psi_{x_0}(x)$$

where

$$\Psi_{x_0}(x)(t) = x_0(0) + \tau \int_0^t F_M[\tilde{x}(s - 1 + \varepsilon x(s))]ds, \quad 0 < t \leq \alpha, \tag{10}$$

and where the tilde above x in the above integrand has the following meaning:

$$\tilde{x}(u) = x(u), \ \text{ if } 0 \leq u \leq \alpha, \ \tilde{x}(u) = x_0(u), \ \text{ if } -\alpha \leq u \leq 0.$$

A solution of (8) is a fixed point of the above Picard Operator.

The space $\mathbf{C}^{1,b}([0,\alpha])$ is equipped with the following metric:

$$\mathbf{d}_1(x, y) = \sup_{0<t<\alpha} |\dot{x}(t) - \dot{y}(t)| = |x - y|_1.$$

We will also use the notation

$$\mathbf{d}_0(x, y) = \sup_{0\leq t\leq\alpha} |x(t) - y(t)| = |x - y|_0.$$

Observe that we always have

$$|x - y|_0 \leq \int_0^\alpha |\dot{x}(s) - \dot{y}(s)|ds \leq \alpha|x - y|_1.$$

A remark on the above notation must be made at this time. It should be noted that $\mathbf{C}^{1,b}([0,\alpha])$ is not in fact a vector space, and so the notation $|x - y|_0$ and $|x - y|_1$ for $x, y \in \mathbf{C}^{1,b}([0,\alpha])$ may be viewed as improper. However, we elect to maintain this notation for ease of reading, justifying its use with the fact that

$$\left(\mathbf{C}^{1,b}([0,\alpha]), \mathbf{d}_0\right) \text{ and } \left(\mathbf{C}^{1,b}([0,\alpha]), \mathbf{d}_1\right)$$

are metric subspaces of the normed spaces

$$\left(\mathbf{C}^0([0,\alpha]), |\cdot|_0\right) \text{ and } \left(\mathbf{C}^1([0,\alpha]), |\cdot|_1\right),$$

respectively.

2.1. *The Step Map*

Instead of studying the above operator Ψ_{x_0}, we introduce the following one:

$$\mathbf{L}_{x_0} : \mathbf{C}^{1,b}_{x_0}([0,\alpha]) \to \mathbf{C}^{1,b}_{x_0}([0,\alpha]),\ x \mapsto \mathbf{L}_{x_0}(x)$$

where

$$\mathbf{L}_{x_0}(x)(t) = x_0(0) + \tau \int_0^t F_M[\tilde{x}^+(s-1+\varepsilon x(s))]ds, \qquad 0 \le t \le \alpha, \tag{11}$$

and where

$$x^+(u) = x_0(0) + \tau \int_0^u F_M[\tilde{x}(s-1+\varepsilon x(s))]ds, \qquad 0 \le u \le 9/10.$$

Roughly speaking, we compute the output of the Picard operator into 2 steps: first on the interval $[0, 9/10]$, and then on the entire interval $[0,\alpha]$. We will now verify that

$$x^+(t) \equiv \Psi_{x_0}(x)(t),\ \text{when } 0 \le t \le 9/10.$$

Recall that the tilde in (11) means

$$\tilde{x}^+(u) = x_0(u) \ \text{ if } \ u < 0,\ \ \tilde{x}^+(u) = x^+(u) \ \text{ if } \ u \ge 0.$$

Observe first that since $0 \le s \le 9/10$, thanks to (5) we have

$$s - 1 + \varepsilon x(s) \le \alpha - 1 + 1/10 < 0,$$

and therefore x^+ can be equivalently defined as

$$x^+(u) = x_0(0) + \tau \int_0^u F_M[x_0(s-1+\varepsilon x(s))]ds, \qquad 0 \le u \le 9/10. \tag{12}$$

Observe that if $x_0 \in \mathbf{C}^{1,b}([-\alpha,0])$ and $x \in \mathbf{C}^{1,b}_{x_0}([0,\alpha])$, both functions x^+ and $\mathbf{L}_{x_0}(x)$ are differentiable. Furthermore, from condition (6) we have that

$$\left|\frac{dx^+}{dt}(t)\right| \le \tau K_0 \le b, \ \text{ and } \ \left|\frac{d\mathbf{L}_{x_0}(x)}{dt}(t)\right| \le \tau K_0 \le b.$$

Therefore both x^+ and $\mathbf{L}_{x_0}(x)$ belong to $\mathbf{C}^{1,b}_{x_0}([0,\alpha])$. The reason why we split the Picard operator Ψ_{x_0} into two parts will be clear later in section 2.3 when we introduce the 'reduced' version of this operator.

We now state the following Theorem.

Theorem 1. *For all $0 < |\varepsilon| \le \varepsilon_0$, $x_0 \in \mathbf{C}^{1,b}([-\alpha,0])$ satisfying (5)(i), and for all $x, y \in \mathbf{C}^{1,b}_{x_0}([0,\alpha])$ satisfying (5)(i), we have*

$$\left|\mathbf{L}_{x_0}(x) - \mathbf{L}_{x_0}(y)\right|_1 \le \gamma|\varepsilon| \cdot |x-y|_1,$$

where $\gamma = \alpha b(\tau K_1 + \tau^2 K_1^2)$.

As a consequence, $\mathbf{L}_{x_0}$ is a contraction, and more precisely there exists $0 < \varepsilon_1 \le \varepsilon_0$ such that for all $0 \le |\varepsilon| < \varepsilon_1$,

$$\left|\mathbf{L}_{x_0}(x) - \mathbf{L}_{x_0}(y)\right|_1 \le \frac{1}{2}|x-y|_1. \tag{13}$$

Observe that the space $\mathbf{C}^{1,b}_{x_0}([0,\alpha])$ is complete with the metric $\mathbf{d}_1$ and for any $x \in \mathbf{C}^{1,b}_{x_0}([0,\alpha])$, the sequence $\mathbf{L}^n_{x_0}(x)$ converges to the fixed point of the operator. We then define the Step Map

$$\phi_\alpha : \mathbf{C}^{1,b}([-\alpha,0]) \to \mathbf{C}^{1,b}([-\alpha,0]),\ z \mapsto \phi_\alpha(z)$$

where

$$\phi_\alpha(z)(t) = P_{\alpha,z}(t+\alpha)$$

and where $P_{\alpha,z}$ is the fixed point of $\mathbf{L}_z$. For two different values of α, say $\alpha_1 < \alpha_2$, each fixed point, although defined on $[0,\alpha_1]$ and $[0,\alpha_2]$ respectively, coincide on $[0,\alpha_1]$. Since the fixed point of $\mathbf{L}_z$ does not depend upon α, we have

$$\frac{\partial \phi_\alpha(z)}{\partial \alpha}(t) = \frac{d\phi_\alpha(z)}{dt}(t). \tag{14}$$

Observe that if there exists an integer $m \geq 1$ and $x_0 \in \mathbf{C}^{1,b}([-\alpha,0])\}$ such that

$$\phi_\alpha^m(x_0) \equiv x_0,$$

the solution $y = y(t)$ of (3) with x_0 as initial condition is periodic, i.e., $y(t+T) = y(t)$, $\forall t \geq -\alpha$, where $T = m\alpha$.

Before proving Theorem 1, we need the following lemma.

Lemma 1. *For all $0 \leq |\varepsilon| \leq \varepsilon_0$, for all $x, y \in \mathbf{C}^{1,b}([0,\alpha])$ satisfying (5)(i), we have*

$$\sup_{0\leq t\leq 9/10} |x^+(t) - y^+(t)| \leq |\varepsilon| b\tau K_1 |x-y|_0.$$

Proof. Let $x, y \in \mathbf{C}^{1,b}([0,\alpha])$ satifying (5)(i). Recall that

$$\begin{aligned} x^+(t) &= x_0(0) + \tau \int_0^t F_M(x_0(s-1+\varepsilon x(s)))ds, \quad \text{and} \\ y^+(t) &= x_0(0) + \tau \int_0^t F_M(x_0(s-1+\varepsilon y(s)))ds \end{aligned} \tag{15}$$

Using (6), (9) and the Mean Value Theorem we have for all $0 \leq t \leq 9/10$,

$$\begin{aligned} \left|\frac{dx^+}{dt}(t) - \frac{dy^+}{dt}(t)\right| &= \tau \Big| F_M(x_0(t-1+\varepsilon x(t))) - F_M(x_0(t-1+\varepsilon y(t)))\Big| \\ &\leq bK_1\tau|\varepsilon| \sup_{0\leq t\leq 9/10} |x(t)-y(t)| \leq bK_1\tau|\varepsilon| \cdot |x-y|_0, \end{aligned}$$

and therefore

$$\sup_{0\leq t\leq 9/10} |x^+(t) - y^+(t)| \leq \int_0^1 \left|\frac{dx^+}{dt}(t) - \frac{dy^+}{dt}(t)\right| dt \leq bK_1\tau|\varepsilon| \cdot |x-y|_0,$$

and Lemma 1 follows. ∎

Proof. [Proof of Theorem 1] Let $s \in [0,\alpha]$. We first write

$$F_M(\tilde{x}^+(s-1+\varepsilon x(s))) - F_M(\tilde{y}^+(s-1+\varepsilon y(s))) = \Delta_1(s) + \Delta_2(s) \tag{16}$$

where

$$\begin{aligned} \Delta_1(s) &= F_M(\tilde{x}^+(s-1+\varepsilon x(s))) - F_M(\tilde{x}^+(s-1+\varepsilon y(s))), \\ \Delta_2(s) &= F_M(\tilde{x}^+(s-1+\varepsilon y(s))) - F_M(\tilde{y}^+(s-1+\varepsilon y(s))). \end{aligned} \tag{17}$$

Recall that

$$\tilde{x}^+(u) = x_0(u), \quad \text{if} \quad u < 0, \quad \tilde{x}^+(u) = x^+(u), \quad \text{if} \quad u \geq 0,$$

and since both x^+ and x_0 are b-Lipshitz, $\tilde{x}^+$ is also b-Lipschitz, i.e., for all $-\alpha \leq u_1 \leq u_2 \leq \alpha$

$$|\tilde{x}^+(u_1) - \tilde{x}^+(u_2)| \leq b|u_1 - u_2|.$$

Due to (6) and the Mean Value Theorem (applied to the function F_M), for all $0 \leq s \leq \alpha$, we have

$$\tau|\Delta_1(s)| \leq b|\varepsilon|\tau K_1|x(s) - y(s)| \ \leq b|\varepsilon|\tau K_1|x - y|_0. \tag{18}$$

Also, we have

$$\tau|\Delta_2(s)| \leq \tau K_1\Big|\tilde{x}^+(s - 1 + \varepsilon y(s))) - \tilde{y}^+(s - 1 + \varepsilon y(s)))\Big|.$$

Moreover, since $0 \leq s \leq \alpha$, from (5) we have

$$s - 1 + \varepsilon y(s) \leq \alpha - 1 + \varepsilon M \leq 9/10,$$

and therefore, if $s - 1 + \varepsilon y(s) < 0$

$$\tilde{x}^+(s - 1 + \varepsilon y(s))) - \tilde{y}^+(s - 1 + \varepsilon y(s))) = x_0(s - 1 + \varepsilon y(s))) - x_0(s - 1 + \varepsilon y(s))) \ = \ 0.$$

Otherwise, if $0 \leq s - 1 + \varepsilon y(s) \leq 9/10$, according to Lemma 1

$$\Big|\tilde{x}^+(s - 1 + \varepsilon y(s))) - \tilde{y}^+(s - 1 + \varepsilon y(s)))\Big| \leq b\tau|\varepsilon|K_1|x - y|_0. \tag{19}$$

Thus, the above inequality holds for all $0 \leq s \leq \alpha$ and we have

$$\tau|\Delta_2(s)| \leq b|\varepsilon|\tau^2 K_1^2|x - y|_0. \tag{20}$$

Now, we write

$$\frac{d\mathbf{L}_{x_0}(x)}{dt}(t) = \tau F_M(\tilde{x}^+(t - 1 + \varepsilon x(t))) \qquad \text{and} \qquad \frac{d\mathbf{L}_{x_0}(y)}{dt}(t) = \tau F_M(\tilde{y}^+(t - 1 + \varepsilon y(t))).$$

From (16), it follows that

$$\Big|\mathbf{L}_{x_0}(x) - \mathbf{L}_{x_0}(y)\Big|_1 \leq \sup_{0 \leq t \leq \alpha}\Big\{\tau\Big|\Delta_1(t)\Big| + \tau\Big|\Delta_2(t)\Big|\Big\}, \tag{21}$$

and due to (18) and (20), we have

$$\Big|\mathbf{L}_{x_0}(x) - \mathbf{L}_{x_0}(y)\Big|_1 \leq b|\varepsilon|(\tau K_1 + \tau^2 K_1^2)|x - y|_0, \tag{22}$$

and since $|x - y|_0 \leq \alpha|x - y|_1$ this ends the proof of Theorem 1. ■

Observe that this operator acts on a space of infinite dimension, which makes the computational approach almost impossible. To this end, we propose a discretized version of this operator, called the Reduced Picard Operator. However, to do so, we will first need to introduce an interpolating operator.

2.2. *Lagrange-Chebyshev interpolating operator*

Let $q > 1$ be an integer and $\mathbb{P}_q[t]$ be the space of polynomial functions in the variable t of degree less than or equal to $q-1$. Let $\ell > 0$ and $\alpha_0 > \alpha$, to be specified shortly. We denote by $\mathrm{L}_\ell([0,\alpha_0])$ the set of real- or vector-valued ℓ-lipschitz functions defined on $[0,\alpha_0]$. More precisely, if h is a component of an element in $\mathrm{L}_\ell([0,\alpha_0])$, we have

$$|h(t_2) - h(t_1)| \leq \ell(t_2 - t_1), \quad 0 \leq t_1 \leq t_2 \leq \alpha_0. \tag{23}$$

In what follows, elements in $\mathrm{L}_b([0,\alpha_0])$ are to be interpolated with Chebyshev polynomials. Therefore, we introduce the following rescaling

$$\mathrm{R} : [0,\alpha_0] \to [-1,1], \; t \mapsto -1 + \frac{2}{\alpha_0}t. \tag{24}$$

This linear map rescales the interval $[0,\alpha_0]$ into the interval $[-1,1]$ in a one-to-one manner. We define the following interpolating operator

$$\mathcal{L}_q : \mathrm{L}_\ell([0,\alpha_0]) \to \mathbb{P}_q, \; \mathbf{f} \mapsto \mathcal{L}_q(\mathbf{f}) \tag{25}$$

where

$$\mathcal{L}_q(\mathbf{f})(t) = \sum_{j=0}^{q-1} c_j (\Lambda_j \circ \mathrm{R})(t) \tag{26}$$

where the Λ_j's ($j = 0, \ldots, q-1$) are the Chebyshev polynomials of the first kind. In other words,

$$\Lambda_j(t) = \cos(j \arccos(t)), \; j = 0, \ldots, q-1,$$

and for $j = 0, \ldots, q-1$ we have

$$c_j = \frac{2}{q} \sum_{k=0}^{q-1} (\mathbf{f} \circ \mathrm{R}^{-1})(u_k)\Lambda_j(u_k), \quad j > 0, \qquad c_0 = \frac{1}{q} \sum_{k=0}^{q-1} (\mathbf{f} \circ \mathrm{R}^{-1})(u_k), \tag{27}$$

where the u_k's are the Chebyshev nodes on $[-1,1]$:

$$u_k = \cos\left(\frac{2k+1}{2q}\pi\right), \quad k = 0, \ldots, q-1.$$

See [Handscomb & Mason, 2002] for more details. The operator $\mathcal{L}_q$ is a linear projection, and for all $h \in \mathrm{L}_\ell([0,\alpha_0])$, $\mathcal{L}_q(h)$ converges uniformly to h on $[0,\alpha_0]$ as q tends to ∞. More precisely, the following Lemma holds.

Lemma 2. *Let $\ell > 0$ and let $z \in L_\ell([0,\alpha_0])$. Then*

$$|\mathcal{L}_q(z) - z|_0 \leq \frac{\alpha(1+\mu_q)}{2q}\ell$$

where

$$\mu_q = \frac{1}{\pi} \sum_{j=0}^{q-1} \cot\left(\frac{(j+1/2)\pi}{2q}\right) = \frac{2}{\pi}\log(q) + 0.9625 + \mathcal{O}(1/q).$$

Moreover we have

$$|\mathcal{L}_q(z)|_0 \leq \mu_q |z|_0 \tag{28}$$

This lemma is a direct consequence of Jackson's Theorem and its Corollary 6.14A in [Handscomb & Mason, 2002].

Remark. Observe that in the algorithm to be presented in the next section (namely the Newton procedure for finding periodic orbits), α is seen as a variable and therefore changes at each time we iterate the corresponding operator. However, for a single set of iterations converging (within the desired precision) to the ensuing solution (i.e., the fixed point of the operator), we want α_0 to be fixed. Assuming that the algorithm is initiated with an approximated value of the period, and therefore an approximated value of α, we choose α_0 to be slightly bigger than α. However, we must verify at each step that the value of α does not exceed α_0.

2.3. *The Reduced Picard Operator*

We define

$$\mathbb{P}_{q-1,b} = \{z \in \mathbb{P}_{q-1}, \ \left|\frac{dz}{dt}(t)\right| \leq b\}.$$

For a given initial condition $x_0 \in \mathbb{P}_{q-1,b}$, the action of the Reduced Picard Operator (RPO) resembles the operator presented in (11), except that the integrand is interpolated on the interval $[0, \alpha_0]$. More precisely we have

$$\Psi_{x_0,q} : \mathbb{P}_{q-1,b} \to \mathbb{P}_{q-1}, \ x \mapsto \Psi_{x_0,q}(x)$$

where

$$\Psi_{x_0,q}(x)(t) = x_0(0) + \tau \int_0^t \mathcal{L}_{q-1}(G_x)(s)ds, \quad 0 \leq t \leq \alpha_0 \tag{29}$$

and where

$$G_x(s) = F_M[\tilde{x}^+(s - 1 + \varepsilon x(s))],$$

recalling that

$$\tilde{x}^+(u) = x^+(u), \ \text{ if } 0 \leq u \leq \alpha, \quad \text{and} \ \tilde{x}^+(u) = x_0(u), \ \text{ if } -\alpha \leq u \leq 0,$$

x^+ being defined in Eq. (12). Since both $\tilde{x}$ and x are b-Lipschitz, it follows that G_x is ℓ-Lipschitz with

$$\ell = K_1 b(1 + \varepsilon b). \tag{30}$$

From now on we will choose q large enough so that for all $x_0 \in \mathbf{C}^{1,b}([-\alpha, 0])$ and for all $x \in \mathrm{L}_b([0, \alpha_0])$ both satisfying (5)(i),

$$\mathbf{d}_0(\mathcal{L}_{q-1}(G_x), G_x) \leq 1/(2\tau).$$

With this latter choice and (9), we have that

$$\left|\frac{d\Psi_{x_0,q}(x)}{dt}(t)\right| \leq \tau|\mathcal{L}_{q-1}G_x(t) - G_x(t)| + \tau|G_x(t)| \leq 1/2 + \tau K_0 \leq b.$$

Therefore, we have that the range of $\Psi_{x_0,q}$ is included in $\mathbb{P}_{q-1,b}$.

Theorem 2. *Let $q > 1$ be an integer and $x_0 \in \mathbf{C}^{1,b}_{x_0}([-\alpha, 0])$. For all $x, \ y \in \mathbb{P}_{q-1,b}$ the Reduced Picard Operator satisfies*

$$\left|\Psi_{x_0,q}(x) - \Psi_{x_0,q}(y)\right|_1 \leq \gamma|\varepsilon| \cdot |x - y|_1, \tag{31}$$

where

$$\gamma = \alpha b \mu_{q-1}(\tau K_1 + \tau^2 K_1^2).$$

As a consequence, there exists $0 < \varepsilon_2 \leq \varepsilon_1$ such that for all $0 < |\varepsilon| \leq \varepsilon_2$, the Reduced Picard Operator is a contraction and the operator $\Psi_{x_0,q}$ admits a fixed point. We then define the Reduced Step Map

$$\phi_{\alpha,q} : \mathbb{P}_{q-1} \to \mathbb{P}_{q-1}, \; z \mapsto \phi_{\alpha,q}(z)$$

where

$$\phi_{\alpha,q}(z)(t) = \mathbf{P}_{\alpha,z,q}(t+\alpha) \tag{32}$$

where the polynomial $\mathbf{P}_{\alpha,z,q}(t)$ is the fixed point of $\Psi_{z,q}$. Observe that for sake of simplicity in the notation, α does not appear in the notations for the Picard (and Reduced) Operator. However, α needs to appear in the notation for both the Step Map and the Reduced Step map, since, α will act as an independent variable of the Newton-Kantorovich operator (to be defined shortly). If we write

$$\mathbf{P}_{\alpha,z,q}(t) = \sum_{j=0}^{q-2} p_j t^j,$$

we then have

$$\phi_{\alpha,q}(z)(t) = \sum_{j=0}^{q-2} w_j t^j, \quad \text{where} \quad \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_{q-2} \end{pmatrix} = \mathbf{T} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{q-2} \end{pmatrix} \tag{33}$$

and where

$$\mathbf{T} = \Big(T_{i,j} \Big)_{1 \leq i,j \leq q-1} \quad \text{with } T_{i,j} = \alpha^{j-i} \binom{j-1}{i-1} \text{ if } j \geq i, \; T_{i,j} = 0 \text{ if } j < i. \tag{34}$$

Also, thanks to (32) we have

$$\frac{\partial \phi_{\alpha,q}(z)(t)}{\partial \alpha} = \frac{d\phi_{\alpha,q}(z)(t)}{dt}(t) = \sum_{j=1}^{q-1} j w_j t^{j-1}. \tag{35}$$

Observe that based upon (31), the choice of ε_1 decreases as q (the number of nodes chosen for the interpolation) increases. Since the choice of q determines how close the interpolation $\mathcal{L}_{q-1}(z)$ of a element $z \in \mathbf{C}^{1,b}([0,\alpha])$ is to z (with respect to the metric $\mathbf{d}_0$), this implies that the more accurate we want our solution to be computed, the smaller ε_2 should be chosen. However, since $\mu_q = \mathcal{O}(\log(q))$, even with a substantially large number of nodes, we still have room to compute the solution of (3) for values of $\varepsilon \geq 1/50$ and with relatively high precision.

Proof. [Proof of Theorem 2] Recall that

$$\begin{aligned} \Psi_{x_0,q}(x)(t) &= x_0(0) + \int_0^t \mathcal{L}_{q-1}(\tau G_x)(s)ds \qquad \text{and} \\ \Psi_{x_0,q}(y)(t) &= x_0(0) + \int_0^t \mathcal{L}_{q-1}(\tau G_y)(s)ds, \end{aligned} \tag{36}$$

where

$$G_x(s) = F_M(\tilde{x}^+(s-1+\varepsilon x(s)))$$

and

$$G_y(s) = F_M(\tilde{y}^+(s-1+\varepsilon y(s))).$$

By linearity of $\mathcal{L}_{q-1}$, we have

$$\frac{d\Psi_{x_0,q}(x)}{dt}(t) - \frac{d\Psi_{x_0,q}(y)}{dt}(t) = \mathcal{L}_{q-1}(\tau G_x - \tau G_y)(t).$$

From (28) in Lemma 2, we have

$$\left|\frac{d\Psi_{x_0,q}(x)}{dt}(t) - \frac{d\Psi_{x_0,q}(y)}{dt}(t)\right| \leq \left|\mathcal{L}_{q-1}(\tau G_x - \tau G_y)(t)\right|_0 \leq \mu_{q-1}\left|(\tau G_x - \tau G_y)(t)\right|_0. \tag{37}$$

Recall that

$$\tau G_x(t) - \tau G_y(t) = \tau F_M(\tilde{x}^+(t-1+\varepsilon x(t))) - \tau F_M(\tilde{y}^+(t-1+\varepsilon y(t))) = \frac{d\mathbf{L}_{x_0}(x)}{dt}(t) - \frac{d\mathbf{L}_{x_0}(y)}{dt}(t),$$

and by Theorem 1, (37) and the above equality, we have

$$\left|\frac{d\Psi_{x_0,q}(x)}{dt}(t) - \frac{d\Psi_{x_0,q}(y)}{dt}(t)\right| \leq b|\varepsilon|\mu_{q-1}(\tau K_1 + \tau^2 K_1^2)|x-y|_0 \leq \alpha b|\varepsilon|\mu_{q-1}(\tau K_1 + \tau^2 K_1^2)|x-y|_1,$$

By taking the supremum over $(0, \alpha)$, this ends the proof of Theorem 2. ■

2.4. *A Newton procedure*

The previous sections concerned the computation of a solution to (7), but not necessarily *periodic* solutions. However, thanks to the above setting, the system under consideration admits a periodic orbit of period T if there exists $\alpha > 1$ and $m \geq 1$ such that $T = m\alpha$ and $x = x(t) \in \mathbf{C}^{1,b}([-\alpha, 0])$ such that

$$\phi_\alpha^m(x) = x.$$

In the computational context, we replace the Step Map by the Reduced Step Map, i.e., we will search for $z = z(t) \in \mathbb{P}_{q-1,b}$ such that

$$\phi_{\alpha,q}^m(z) = z. \tag{38}$$

To initiate our search, we need an additional restriction. Observe that if $x = x(t)$ defined on $\mathbb{R}$ is a periodic solution of our system, then for all $\beta \in \mathbb{R}$, $x(t+\beta)$ is also a periodic orbit with the same period. To be able to find an isolated periodic orbit, we request the solution to satisfy $x(0) = \beta$, where β has to be chosen in the interior of the (a-priori) range of the orbit. We denote by

$$\mathbb{P}_{q-1}^* = \{y \in \mathbb{P}_{q-1} \text{ with } y(0) = \beta\}.$$

In the notation below, we identify each polynomial with the row-vector with entries representing its coefficients. More precisely we introduce the following isomorphisms:

$$\begin{aligned} \mathbf{J}: \ \mathbb{R}^{q-1} &\to \mathbb{P}_{q-1}, \ (a_0, \ldots, a_{q-2}) \mapsto z = z(t) = \sum_{j=0}^{q-2} a_j t^j \\ \mathbf{J}^*: \ \mathbb{R}^{q-2} &\to \mathbb{P}_{q-1}^*, \ (a_1, \ldots, a_{q-2}) \mapsto z^* = z^*(t) = \beta + \sum_{j=1}^{q-2} a_j t^j. \end{aligned} \tag{39}$$

We write

$$\mathbb{S}_q = [11/10, 9/5] \times \mathbb{R}^{q-2} \subset \mathbb{R}^{q-1}, \quad \text{and} \quad \mathrm{a}_0 = (\alpha, \mathrm{a}^*) \in \mathbb{S}_q.$$

Searching for a solution $z^* \in \mathbb{P}^*_{q-1}$ that satisfies (38) amounts to looking for a zero of the following map

$$\mathbf{G} : \mathbb{S}_q \times \underbrace{\mathbb{R}^{q-1} \times \cdots \times \mathbb{R}^{q-1}}_{m-1\ times} \to \underbrace{\mathbb{R}^{q-1} \times \cdots \times \mathbb{R}^{q-1}}_{m\ times},$$
$$\mathbf{a} = (\mathrm{a}_0, \mathrm{a}_1, \ldots, \mathrm{a}_{m-1}) \mapsto \Big(G_0(\mathbf{a}), \ldots, G_{m-1}(\mathbf{a})\Big) \tag{40}$$

where

$$\begin{aligned} G_0(\mathbf{a}) &= \mathbf{J}^{-1}\Big(\phi_{\alpha,q}(\mathbf{J}^*(\mathrm{a}^*)) - \mathbf{J}(\mathrm{a}_1)\Big), \quad G_{m-1}(\mathbf{a}) = \mathbf{J}^{-1}\Big(\phi_{\alpha,q}(\mathbf{J}(\mathrm{a}_{m-1})) - \mathbf{J}^*(\mathrm{a}^*)\Big), \\ &\text{and} \\ G_j(\mathbf{a}) &= \mathbf{J}^{-1}\Big(\phi_{\alpha,q}(\mathbf{J}(\mathrm{a}_j)) - \mathbf{J}(\mathrm{a}_{j+1})\Big), \quad \text{for} \ \ j = 1, \ldots, m-2. \end{aligned} \tag{41}$$

Zeroes of the map $\mathbf{G}$ are in 1-1 correspondence with fixed points of the Newton-like operator

$$\mathbf{H} : \mathbb{S}_q \times \underbrace{\mathbb{R}^{q-1} \times \cdots \times \mathbb{R}^{q-1}}_{m-1\ times} \to \mathbb{S}_q \times \underbrace{\mathbb{R}^{q-1} \times \cdots \times \mathbb{R}^{q-1}}_{m-1\ times}, \quad \mathbf{z} \mapsto \mathbf{z} - \mathbf{B} \circ \mathbf{G}(\mathbf{z}), \tag{42}$$

where $\mathbf{B}$ is an operator to be determined shortly. Indeed, we need $\mathbf{H}$ to be a contraction. This will be satisfied if we can find $\mathbf{B}$ close enough to $\mathbf{A}^{-1}$ where $\mathbf{A}$ is itself close enough to the differential of $\mathbf{G}$ at the point $\mathbf{z}$. Therefore, we need to estimate the partial derivatives of the Reduced Step Map with respect to the variables α, a^*, $\mathrm{a}_1, \ldots, \mathrm{a}_{m-2}$, and a_{m-1}. Observe that the derivative of the Reduced Step Map with respect to α is given in (35). In the next section the choice of $\mathbf{A}$ will be determined from the variational equation.

2.5. *Variational Equation*

The operator $\mathbf{A}$ is computed by solving the Variational Equation associated with (3), when $\varepsilon = 0$. More precisely we append the Variational Equation associated to equation (3) when $\varepsilon = 0$ and study the Reduced Picard Operator associated to this (extended) equation. With this choice of $\mathbf{A}$, we verify, a-posteriori, that $\mathbf{H}$ is a contraction.

Recall that for a given initial condition $y \in \mathbb{P}^*_{q-1}$ or $y \in \mathbb{P}_{q-1}$, for $t > 0$, our search is for a piecewise polynomial solution of the equation

$$\dot{x} = \tau \mathcal{L}_{q-1}\Big(F[x(t - 1 + \varepsilon x(t))]\Big), \quad t > 0 \tag{43}$$

where

$$x(u) = y(u), \quad \text{if} \ -\alpha \le u \le 0.$$

We first compute the partial derivative of the solution above with respect to any variable ξ parameterizing the initial condition y (defined on the interval $[-\alpha, 0]$). In the case $y \in \mathbb{P}^*_{q-1}$ we write

$$y(t) = \beta + \sum_{j=1}^{q-2} y_j t^j,$$

and the y_j's, $j = 1, \ldots, q-2$ are the variables parameterizing y. If $\xi = y_j$ for some $1 \le j \le q-2$, we have that

$$\frac{\partial y}{\partial \xi}(t) = t^j.$$

In the case $y \in \mathbb{P}_{q-1}$ we write

$$y(t) = \sum_{j=0}^{q-2} y_j t^j,$$

and the y_j's, $j = 0, \dots, q-2$ are the variables parameterizing y. If $\xi = y_j$ for some $0 \le j \le q-2$, we have that

$$\frac{\partial y}{\partial \xi}(t) = t^j.$$

From Eq. (43), and since $\mathcal{L}_{q-1}$ is a linear operator, it follows that, for $t > 0$, we have

$$\frac{\partial \dot{x}}{\partial \xi} = \tau \mathcal{L}_{q-1}\Big(F'[x(t-1+\varepsilon x(t))] B_\xi(t) \Big),$$

where

$$B_\xi(t) = \frac{\partial x}{\partial \xi}(t-1+\varepsilon x(t)) + \varepsilon \dot{x}(t-1+\varepsilon x(t)) \frac{\partial x}{\partial \xi}(t). \tag{44}$$

Observe the second term in (44) may be discontinuous in t leading to a possible Gibbs-like effect in the interpolation. Therefore, we set $\varepsilon = 0$ in the second term of (44), yielding

$$\frac{\partial \dot{x}}{\partial \xi} = \tau \mathcal{L}_{q-1}\Big(F'[x(t-1+\varepsilon x(t))] \frac{\partial x}{\partial \xi}(t-1+\varepsilon x(t)) \Big).$$

Introducing

$$u = (u_0, \dots, u_{q-2}) = \left(\frac{\partial x}{\partial y_0}, \dots, \frac{\partial x}{\partial y_{q-2}} \right),$$

we aim to solve the following Initial Value Problem

$$\begin{cases} \dot{x} = \tau \mathcal{L}_{q-1}\Big(F[x(t-1+\varepsilon x(t))] \Big), \\ \dot{u}_j = \tau \mathcal{L}_{q-1}\Big(F'[x(t-1+\varepsilon x(t))] u_j(t-1+\varepsilon x(t)) \Big), \\ x(t) = y(t) \quad \& \quad u_j(t) = \dfrac{\partial y}{\partial y_j}(t) = t^j, \qquad \text{if} \quad -\alpha \le t \le 0, \quad j = 0, \dots, q-2. \end{cases} \tag{45}$$

To estimate the solution of (45), we proceed in the same manner as when solving Eq. (4), using the Reduced Picard iteration process described earlier in this section. More precisely, for a given initial condition $y \in \mathbf{C}^{1,b}([-\alpha, 0])$, we consider the iteration of the Reduced Picard operator

$$(x, u) \mapsto \mathbf{\Psi}_{y,q}(x, u) = (\Psi_{y,q}(x), \hat{\Psi}_{y,q}(x, u)), \tag{46}$$

with

$$\hat{\Psi}_{y,q}(x, u) = (\hat{\Psi}_{y,q,1}(x, u), \dots, \hat{\Psi}_{y,q,q-2}(x, u)),$$

where from (45)

$$\begin{cases} \Psi_{y,q}(x) = y(0) + \tau \displaystyle\int_0^t \mathcal{L}_{q-1}\Big(F_M[\tilde{x}^+(s-1+\varepsilon x(s)))] \Big) ds, \\ \hat{\Psi}_{y,q,j}(x, u) = \dfrac{\partial y}{\partial y_j}(0) + \tau \displaystyle\int_0^t \mathcal{L}_{q-1}\Big(F'[\tilde{x}^+(s-1+\varepsilon x(s))] \tilde{u}_j^+(s-1+\varepsilon x(s)) \Big) ds, \\ \qquad j = 0, \dots, q-2, \end{cases} \tag{47}$$

where

$$\begin{cases} x^+(t) = y(0) + \tau \displaystyle\int_0^t F_M[y(s-1+\varepsilon x(s))]ds, & 0 \le t \le 9/10 \\ u_j^+(t) = \dfrac{\partial y}{\partial y_j}(0) + \tau \displaystyle\int_0^t F'_M[y(s-1+\varepsilon x(s))]\tilde{u}_j(s-1+\varepsilon x(s))ds, & 0 \le t \le 9/10 \\ \qquad j = 0, \dots, q-2 \end{cases} \tag{48}$$

and where

$$x(t) = y(t) \quad \& \quad u_j(t) = \frac{\partial y}{\partial y_j}(t) = t^j \quad \text{if } -\alpha \le t \le 0, \quad j = 0, \dots, q-2.$$

We also recall that

$$\tilde{x}^+(t) = y(t) \text{ if } t < 0, \quad \tilde{x}^+(t) = x^+(t) \text{ if } t \ge 0,$$

and similarly

$$\tilde{u}_j^+(t) = \frac{\partial y}{\partial y_j}(t) = t^j \quad \text{if } t < 0, \; \tilde{u}_j^+(t) = u_j^+(t) \text{ if } t \ge 0 \quad j = 0, \dots, q-2.$$

For a given initial condition $z \in \mathbb{P}_{q-1}$ we denote by

$$(\mathbf{P}_{\alpha,q,z}, \mathbf{U}_{\alpha,q,z}), \quad \text{where} \quad \mathbf{U}_{\alpha,q,z} = (U_{\alpha,z,0}, \dots, U_{\alpha,z,q-2}),$$

the fixed point of the map $\mathbf{\Psi}_{z,q}$. Observe that $\mathbf{P}_{\alpha,q,z}$ is the fixed point of $\Psi_{\alpha,q}(z)$ and is defined in (32). We have

$$\mathbf{J}^{-1}(\phi_{\alpha,q}(z)) = \mathbf{T}\left(\mathbf{J}^{-1}\left(\mathbf{P}_{\alpha,q,z}\right)\right),$$

where $\mathbf{T}$ is defined in (34). Therefore, the differential of the Step Map (modulo the isomorphism defined in (39)) is represented by the matrix

$$\mathbf{A}(\alpha, z) = \mathbf{T}\mathbf{U}_{\alpha,q,z}. \tag{49}$$

2.6. *Construction of A*

We write

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_0^* & -\mathrm{Id}_{q-1} & \mathbf{0}_{q-1} & \mathbf{0}_{q-1} & \cdots & \cdots & \mathbf{0}_{q-1} \\ \mathbf{0}^*_{q-1,1} & \mathbf{A}_1 & -\mathrm{Id}_{q-1} & \mathbf{0}_{q-1} & \cdots & \cdots & \mathbf{0}_{q-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}^*_{q-1,j} & \cdots & \mathbf{0}_{q-1} & \mathbf{A}_j & -\mathrm{Id}_{q-1} & \cdots & \mathbf{0}_{q-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}^*_{q-1,m-2} & \cdots & \cdots & \cdots & \mathbf{0}_{q-1} & \mathbf{A}_{m-2} & -\mathrm{Id}_{q-1} \\ -\mathrm{Id}^*_{q-1} & \mathbf{0}_{q-1} & \cdots & \cdots & \cdots & \mathbf{0}_{q-1} & \mathbf{A}_{m-1} \end{pmatrix}$$

where $\mathbf{0}_{q-1}$ and Id_{q-1} are respectively the zero matrix and the identity matrix, both of dimension $q-1$, and

$$\mathbf{A}_k = \mathbf{A}(\alpha, \mathbf{J}(\mathrm{a}_k)), \quad k = 1, \dots, m-1.$$

Furthermore, every column of the matrix $\mathbf{A}_0^*$ coincides with those of $\mathbf{A}(\alpha, \mathbf{J}(\mathrm{a}_0))$, except for the first column of $\mathbf{A}_0^*$. This latter is given by the vector

$$\mathbf{J}^{-1}\left(\frac{\partial\phi_{\alpha,q}(\mathbf{J}^*(\mathrm{a}^*))}{\partial\alpha}\right) = \begin{pmatrix} w_{0,1} \\ 2w_{0,2} \\ \vdots \\ (q-2)w_{0,q-2} \\ 0 \end{pmatrix}, \quad \text{where } \mathbf{J}^*\left(\begin{pmatrix} w_{0,1} \\ w_{0,2} \\ \vdots \\ w_{0,q-2} \end{pmatrix}\right) = \phi_{\alpha,q}\Big(\mathbf{J}^*(\mathrm{a}^*)\Big).$$

Similarly, for each integer $1 \le j \le m-2$, every column of the matrix $\mathbf{0}_{q-1,j}^*$ coincides with those of $\mathbf{0}_{q-1}$, except for the first column of $\mathbf{0}_{q-1,j}^*$. This latter is given by the vector

$$\mathbf{J}^{-1}\left(\frac{\partial\phi_{\alpha,q}(\mathbf{J}(\mathrm{a}_j))}{\partial\alpha}\right) = \begin{pmatrix} w_{j,1} \\ 2w_{j,2} \\ \vdots \\ (q-2)w_{j,q-2} \\ 0 \end{pmatrix}, \quad \text{where } \mathbf{J}\left(\begin{pmatrix} w_{j,0} \\ w_{j,1} \\ \vdots \\ w_{j,q-2} \end{pmatrix}\right) = \phi_{\alpha,q}\Big(\mathbf{J}(\mathrm{a}_j)\Big),$$

Finally, $-\mathrm{Id}_{q-1}^*$ coincides with $-\mathrm{Id}_{q-1}$, except that the first column is given by the vector

$$\mathbf{J}^{-1}\left(\frac{\partial\phi_{\alpha,q}(\mathbf{J}(\mathrm{a}_{m-1}))}{\partial\alpha}\right) = \begin{pmatrix} w_{m-1,1} \\ 2w_{m-1,2} \\ \vdots \\ (q-2)w_{m-1,q-2} \\ 0 \end{pmatrix}, \quad \text{where } \mathbf{J}\left(\begin{pmatrix} w_{m-1,0} \\ w_{m-1,1} \\ \vdots \\ w_{m-1,q-2} \end{pmatrix}\right) = \phi_{\alpha,q}\Big(\mathbf{J}(\mathrm{a}_{m-1})\Big).$$

3. Implementation and a-posteriori check

The Newton-like operator defined above is implemented in a classical manner. We first need to find an 'initial guess', that is, in our context, an element of $z \in \mathbb{P}_{q-1}^*$ for which

$$\phi_{\alpha,q}(z) \sim z.$$

In the next subsection, we describe how to construct a set of initial guesses for different values of the parameters.

3.1. *Search for initial guess*

Finding such a guess could be challenging. In the present situation, the period T (which is unknown) is indeed represented by two quantities: α and the integer m where $T = m\alpha$. This makes our search substantially more complicated. To overcome this difficulty we initiate our search for periodic solutions of Eq. (3), by searching first for periodic solutions of the same equation when $\varepsilon = 0$, using a Fourier series ansatz. In other words, we consider

$$\dot{x}(t) = \tau\left[x(t-1) - x^3(t-1)\right]. \tag{50}$$

Since our search is for periodic orbits, we do not need to specify the 'past' function x_0 as we did in (3).

3.1.1. *Formulating Maps*

We formulate a Fourier series ansatz to be plugged into equation (50). More precisely, we write

$$\tilde{x}(t) = \sum_{n\in\mathbb{Z}} a_n e^{\omega_n t} \tag{51}$$

for $t \geq -1$, where $\omega_n = \dfrac{2\pi i n}{T}$ and T is the (unknown) period of $\tilde{x}$. We compute

$$\dot{\tilde{x}}(t) = \sum_{n\in\mathbb{Z}} \omega_n a_n e^{\omega_n t},$$
$$\tilde{x}(t-1) = \sum_{n\in\mathbb{Z}} a_n e^{-\omega_n} e^{\omega_n t},$$
$$\tilde{x}^3(t-1) = \sum_{n\in\mathbb{Z}} \left(\sum_{m\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} a_{n-m} a_{m-\ell} a_\ell \right) e^{-\omega_n} e^{\omega_n t}.$$

See, for instance [Castelli *et al.*, 2018; Jaquette *et al.*, 2017; Jones, 1962a,b] and also [van den Berg *et al.*, 2022] for an illustration of this method in the case of the Mackey-Glass equation [Mackey & Glass, 1977]. The advantage of this method is that we do not need to split the orbit into m trajectories as proposed in the procedure described in the previous section. Plugging the above three series into (50) and bringing everything to one side of the equation yields

$$\sum_{n\in\mathbb{Z}} \left(\omega_n a_n - \tau e^{-\omega_n} \left(a_n - \sum_{m\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} a_{n-m} a_{m-\ell} a_\ell \right) \right) e^{-\omega_n t} = 0. \tag{52}$$

Hence, we are looking for a sequence $a := \{a_n\}_{n\in\mathbb{N}}$ and a real number $T > 0$ such that (52) holds for all t. To do this, we will encode equation (52) as follows. For all $n \in \mathbb{Z}$, define $F_n : \ell^\infty \times \mathbb{R} \to \mathbb{R}$ by

$$F_n(a,T) = \omega_n a_n - \tau e^{-\omega_n} \left(a_n - (a * a * a)_n \right), \tag{53}$$

where $* : \ell^\infty \times \ell^\infty \to \ell^\infty$ is the (extended) Cauchy product (sometimes called the convolution product) defined term-wise by

$$(a * b)_n = \sum_{m\in\mathbb{Z}} a_{n-m} b_m,$$

and ℓ^∞ is the set of all bounded sequences of real numbers. We notice that a pair $(a,T) \in \ell^\infty \times \mathbb{R}$ satisfies equation (52) for all t if and only if $F_n(a,T) = 0$ for all $n \in \mathbb{Z}$. Hence, we will consider the map $F : \ell^\infty \times \mathbb{R} \to \ell^\infty$ defined by $F(a,T) = \{F_n(a,T)\}_{n\in\mathbb{Z}}$.

We again impose an additional condition on the value at $t = 0$ of the periodic orbit(s) we are searching for. Namely, we require

$$\tilde{x}(0) = \sum_{n\in\mathbb{Z}} a_n = \beta,$$

where β is a value in the range of the solutions of (50). We introduce the map $G_0 : \ell^\infty \times \mathbb{R} \to \mathbb{R}$ defined by

$$G_0(a,T) = \sum_{n\in\mathbb{Z}} a_n - \beta,$$

and form the desired square operator $G : \ell^\infty \times \mathbb{R} \to \ell^\infty \times \mathbb{R}$,

$$G(a,T) = \begin{pmatrix} F(a,T) \\ G_0(a,T) \end{pmatrix}.$$

We now have that $G(a,T) = 0$ if and only if (a,T) satisfies equation (52) and $\tilde{x}(0) = \beta$.

3.1.2. *Newton's Method*

We will use Newton's method to find zeroes of G. To do this, we must first compute the derivative of G at an arbitrary point (a,T). We can express $DG_{(a,T)}$ in terms of its action on a vector in $\ell^\infty \times \mathbb{R}$. If $h \in \ell^\infty$

and $h^* \in \mathbb{R}$, then

$$DG_{(a,T)} \begin{pmatrix} h \\ h^* \end{pmatrix} = \begin{pmatrix} \dfrac{\partial F}{\partial a} & \dfrac{\partial F}{\partial T} \\ \dfrac{\partial G_0}{\partial a} & \dfrac{\partial G_0}{\partial T} \end{pmatrix} \begin{pmatrix} h \\ h^* \end{pmatrix},$$

where

$$\left(\frac{\partial F}{\partial a} h\right)_n = \omega_n h_n - \tau e^{-\omega_n} \left(h_n - 3(a * a * h)_n\right),$$

$$\left(\frac{\partial F}{\partial T} h^*\right)_n = -\frac{h^* \omega_n}{T} \left[a_n + \tau e^{-\omega_n} \left(a_n - (a * a * a)_n\right)\right],$$

$$\frac{\partial G_0}{\partial a} h = \sum_{n \in \mathbb{Z}} h_n,$$

$$\frac{\partial G_0}{\partial T} h^* = 0.$$

We now introduce the operator

$$\mathcal{H} : \ell^\infty \times \mathbb{R} \to \ell^\infty \times \mathbb{R}, \quad x \mapsto \mathbf{x} - \left(DG(\mathbf{x})\right)^{-1} G(\mathbf{x}).$$

We note that a fixed point of $\mathcal{H}$ corresponds to a zero of G.

3.1.3. *Strategy*

In practice we cannot computationally find the zeros of a map defined on an infinite-dimensional space, but we can instead search for zeros of the truncated map $G^N : \mathbb{R}^{2N+2} \to \mathbb{R}^{2N+2}$ defined by

$$G^N(a,T) = \begin{pmatrix} F^N(a,T) \\ G_0^N(a,T) \end{pmatrix}$$

where

$$F^N(a,T) = -\beta + \{F_n(a,T)\}_{n=-N}^{N}$$

and

$$G_0^N(a,T) = \sum_{n=-N}^{N} a_n.$$

We now employ Newton's method on the map G^N to search for zeros, which may in turn approximate zeros of G, given that the solution found is not spurious. Our interests are with values of τ for which the corresponding dynamics show some form of complexity; see for instance [Mireles-James *et al.*, 2021; Groothedde & Mireles-James, 2017; van den Berg *et al.*, 2022] for more details and discussion. To do this, we must find a suitable initial guess to run the algorithm, that is the first $2N+1$ Fourier coefficients and period approximating a zero of G^N. In the case of the Cubic Ikeda map, the convergence of the above algorithm is observed for several initial guesses. Some are leading to an attracting periodic orbit and are fairly easy to find. This is quite typical when τ is relatively small, say $\tau < 1.5$. The initial guesses used for those small values of τ can be used again, by performing a continuation of each periofic orbit for larger and larger value of τ. In particular, for the study case under consideration, we retrieve periodic orbits that are not Lyapunov stable. Not surprisingly, sometimes the algorithm does not converge when we encounter a bifurcation point. (Overcoming this issue amounts to studying in a more general context part of the bifurcations of the system, and is an avenue for future work.) In the case where the delay is constant, see

[Lentjes *et al.*, 2023a,b; Roose & Szalai, 2007] for more details, discussion, and illustrations through other models. Figure (1) below displays a selection of periodic orbits obtained using this approach for several values of τ. For instance, when $\tau = 1.672$, the Lissajou plot displayed in Figure (1) reveals the co-existence of a periodic orbit (bottom left) with more complicated dynamics. More orbits are displayed in Figures (4), (3), and (2). Observe, finally, that for values of τ larger than 1.72, no solution seems to be bounded. In fact, these solutions blow up quite quickly (see [van den Berg *et al.*, 2022] for more discussion).

3.2. *Fixing the tolerance and a-posteriori check*

Recall that, for a given initial condition defined on $[-\alpha, 0]$, the solution of the DDE under consideration is the fixed point of the Picard Operator defined in Section 3. Since this operator is approximated by the Reduced Picard Operator, we need to estimate the difference between the fixed point of the latter operator (computed within the imposed tolerance) and the fixed point of the former operator (the 'true' solution). This difference is the error introduced each time we compute the step map.

We state the following proposition.

Proposition 1. *Let $\nu > 0$ and let $x_0 \in \mathbf{C}^{1,b}([-\alpha, 0])$ and $\mathbf{P}_{\alpha,x_0}$ satisfy*

$$\mathbf{P}_{\alpha,x_0}(t+\alpha) = \phi_\alpha(x_0)(t).$$

There exist $q_1 > 1$ and $\varepsilon_1 > 0$ such that, for all $y \in \mathbb{P}_{q-1,b}[t]$ and for all $0 \le |\varepsilon| \le \varepsilon_1$, there exists $n_1 > 1$ such that, for all $n \ge n_1$,

$$\mathbf{d}_1\Big(\mathbf{P}_{\alpha,x_0}, \Psi^n_{x_0,q_1}(y)\Big) \le \nu.$$

Proof. Let $\nu > 0$ and $x_0 \in C^{1,b}([-\alpha, 0])$ be given. Using Lemma 2, we choose an integer $q_1 > 1$ such that, for all $\omega \in \mathbf{C}^{1,b}([0, \alpha])$ and for all $q \ge q_1$,

$$\mathbf{d}_1\Big(\mathcal{L}_{q-1}(G_\omega), G_\omega\Big) \le \nu/(4\tau). \tag{54}$$

Recall that

$$G_\omega(s) = F_M(\tilde{\omega}^+(s-1+\varepsilon\omega(s)),$$

where

$$\tilde{\omega}^+(u) = \omega^+(u), \text{ if } \ u \ge 0, \quad \tilde{\omega}^+(u) = x_0(u), \text{ if } u \le 0,$$

and

$$\omega^+(t) = \omega(0) + \int_0^t F_M(x_0(s-1+\varepsilon\omega(s))ds, \quad 0 \le t \le 9/10.$$

From Eq. (54), we have

$$\mathbf{d}_1\Big(\mathbf{L}_{x_0}(\omega), \Psi_{x_0,q}(\omega)\Big) \le \nu/4. \tag{55}$$

Let $y \in \mathbb{P}_{q-1,b}[t]$ and construct the sequence of functions

$$\{y_n\}_{n=0}^\infty \subset \mathbb{P}_{q-1,b}[t], \quad \text{such that} \ \ y_{n+1}(t) = \Psi_{x_0,q}(y_n)(t) = \Psi^n_{x_0,q}(y_0)(t), \quad y_0 := y.$$

Thanks to Theorem 2, the above sequence converges and therefore there exists an integer $n_1 \ge 1$ such that

$$\mathbf{d}_1\Big(\Psi^m_{x_0,q}(y_n), y_n\Big) \le \nu/4, \quad \forall n \ge n_1, \quad \forall m \ge 1. \tag{56}$$

By definition we have

$$\mathbf{L}_{x_0}(\mathbf{P}_{\alpha,x_0}) = \mathbf{P}_{\alpha,x_0},$$

and for all $n \geq n_1$ we also have

$$\mathbf{d}_1\Big(\mathbf{P}_{\alpha,x_0}, y_n\Big) = \mathbf{d}_1\Big(\mathbf{L}_{x_0}(\mathbf{P}_{\alpha,x_0}), y_n\Big) \tag{57}$$

$$\leq \mathbf{d}_1\Big(\mathbf{L}_{x_0}(\mathbf{P}_{\alpha,x_0}), \mathbf{L}_{x_0}(y_n)\Big)$$

$$+ \mathbf{d}_1\Big(\mathbf{L}_{x_0}(y_n), \Psi_{x_0,q}(y_n)\Big) + \mathbf{d}_1\Big(\Psi_{x_0,q}(y_n), y_n\Big).$$

From (13) we have

$$\mathbf{d}_1\Big(\mathbf{L}_{x_0}(\mathbf{P}_{\alpha,x_0}), \mathbf{L}_{x_0}(y_n)\Big) \leq \frac{1}{2}\mathbf{d}_1\Big(\mathbf{P}_{\alpha,x_0}, y_n\Big). \tag{58}$$

Furthermore, from (55) we have

$$\mathbf{d}_1\Big(\mathbf{L}_{x_0}(y_n), \Psi_{x_0,q}(y_n)\Big) \leq \nu/4. \tag{59}$$

Finally, combining (56), (57), (58) and (59), for all $n \geq n_1$ we get

$$\mathbf{d}_1\Big(\mathbf{P}_{x_0}, y_n\Big) \leq \frac{1}{2}\mathbf{d}_1\Big(\mathbf{P}_{x_0}, y_n\Big) + \nu/4 + \nu/4,$$

and therefore

$$\mathbf{d}_1\Big(\mathbf{P}_{x_0}, y_n\Big) \leq \nu,$$

ending the proof of Proposition 1. ■

Thanks to the above proposition, for a given intial condition x_0, we can verify the accuracy of each solution we found in the following manner. For $y \in \mathbb{P}_{q-1,b}[t]$, we first choose q_1 such that (54) holds and $n_1 > 1$ such that (56) holds. Recall that for each given initial condition x_0, we aim to solve equation (7). We then compute

$$\mathcal{E}(x_0, q, t) = \dot{y}_m(t) - \tau F_M(\tilde{y}_m(t - 1 + \varepsilon y_m(t)))$$

(where $\tilde{y}_m(u) = x_0(u)$ if $u < 0$, and $\tilde{y}_m(u) = y_m$ if $u \geq 0$), and verify that

$$\Big|\mathcal{E}(x_0, q, t)\Big| \leq \nu, \quad 0 \leq t \leq \alpha.$$

3.3. *Illustration and implementation*

Recall that each periodic orbit found by the method described in Section 3.1 is represented by a truncated Fourier series. To apply the Newton-Kantorovitch algorithm described in Section 2.4, we choose an integer m such that

$$1/10 \leq T/m \leq 9/5,$$

where T is the period of the truncated Fourier series. We choose a value β in the range of this truncated series $\tilde{x}$ and choose t_β such that $\tilde{x}(t_\beta) = \beta$. Define $\tilde{y}(t) = \tilde{x}(t + t_\beta)$ and $\alpha = T/m$. We interpolate $\tilde{y}$ on the interval $[-\alpha, 0]$ with a polynomial $\mathrm{P} = \mathrm{P}(t)$ of degree q_1 (where q_1 is given in Proposition 1). This polynomial becomes our initial guess to start the algorithm described in Section 2.4 for $\varepsilon > 0$. We illustrate our approach with plots of the periodic solutions for different values of the parameters τ and ε, (see figures below). Each plot is displayed (left) in Lissajou, i.e., in the $(x(t), x(t-1))$-space, and (right) in the $(t, x(t))$ space. Starting from each initial guess, we find a continuation of those periodic orbits not only for other values of τ (i.e., up to $\tau \sim 1.68$), but also for different values of ε.

In our computation, we fixed the tolerance to be $\nu = 10^{-7}$. This latter choice is fulfilled by taking 10 nodes for the interpolation operator, i.e., $q = 10$. Our approach does not take rigorous estimates into consideration, (this latter aspect is left for several future projects including the use of computer-assisted proof). However, for some specific cases, we check the accuracy of our result by redoing the same computation with more nodes, where we observe the same figures.

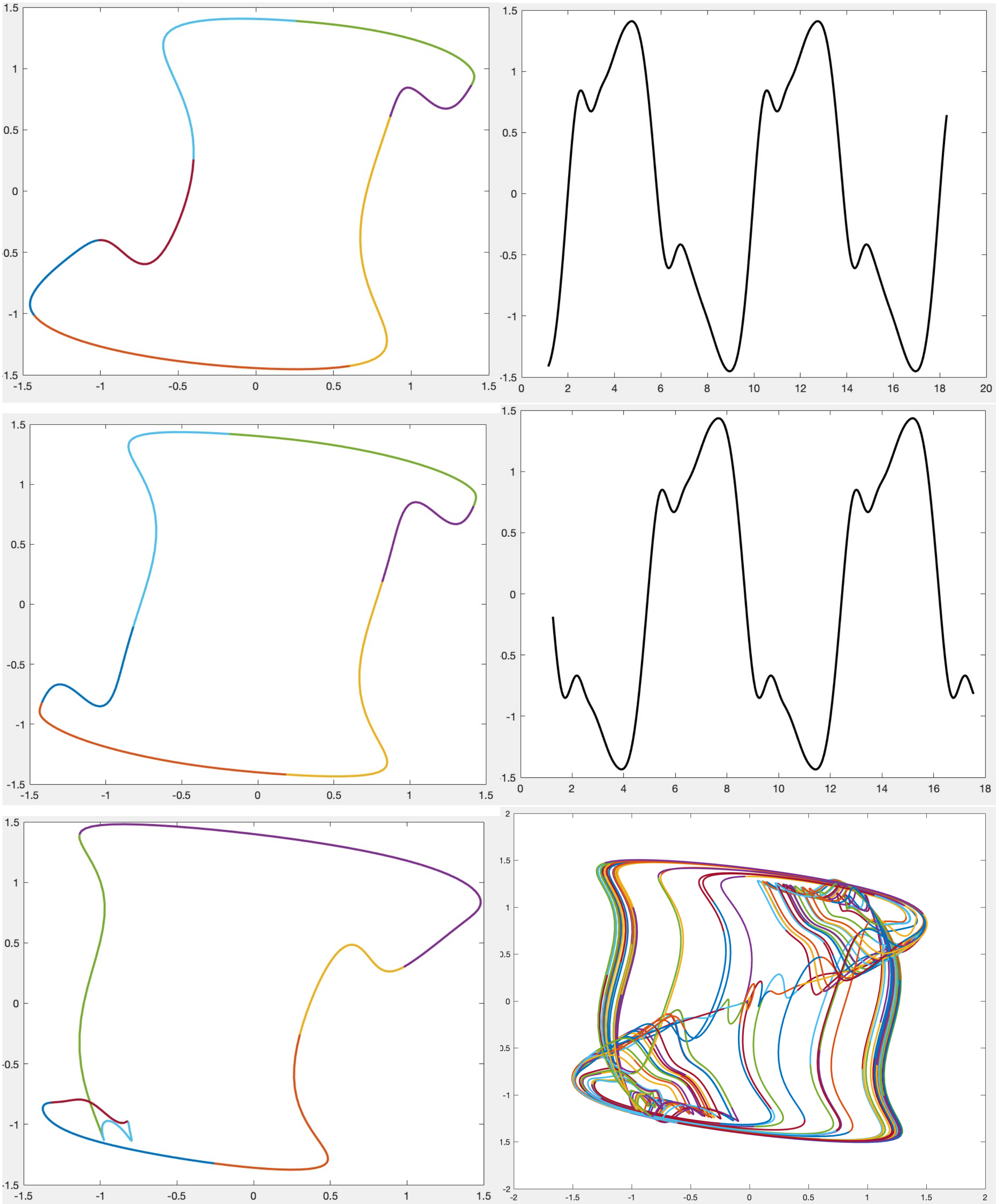

Fig. 1. Periodic orbits when $\varepsilon = 0$. Top: $\tau = 1.567$ Center: $\tau = 1.65$. Bottom left, a periodic orbit in Lissajou with $\tau = 1.6725$. Bottom right: a typical orbit in Lissajou when $\tau = 1.6725$.

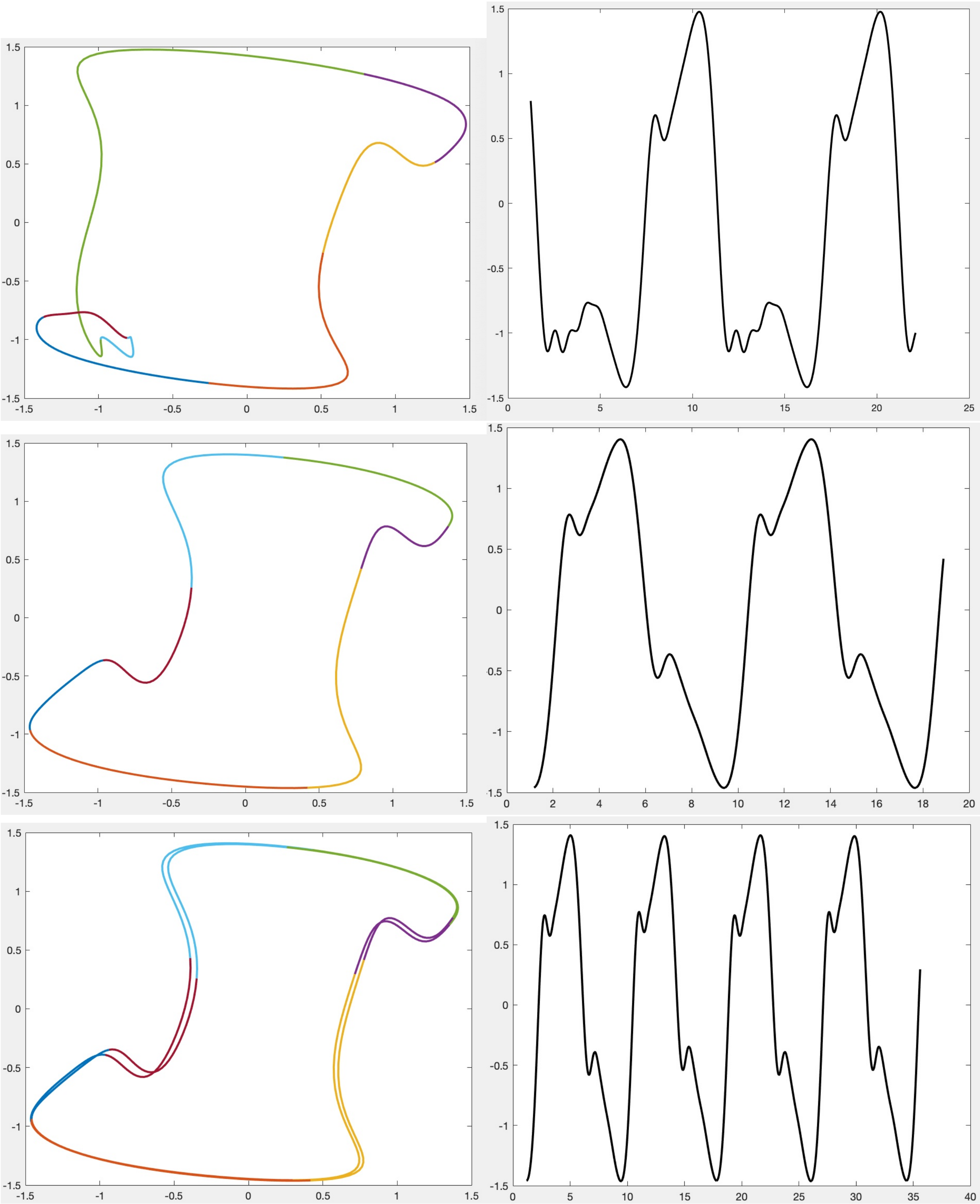

Fig. 2. Top left and right: a periodic orbit for $\varepsilon = 0.0205$ and $\tau = 1.65$, with period $T \sim 11.62$. Middle left and right: a periodic orbit for $\varepsilon = 1.5155$ and $\tau = 0.041$, with period $T \sim 8.26$. Bottom left and right: another periodic orbit for the same value of ε, τ as above with $T \sim 18.53$. Observe this third periodic orbit may be the continuation of the second orbit after a period-doubling bifurcation.

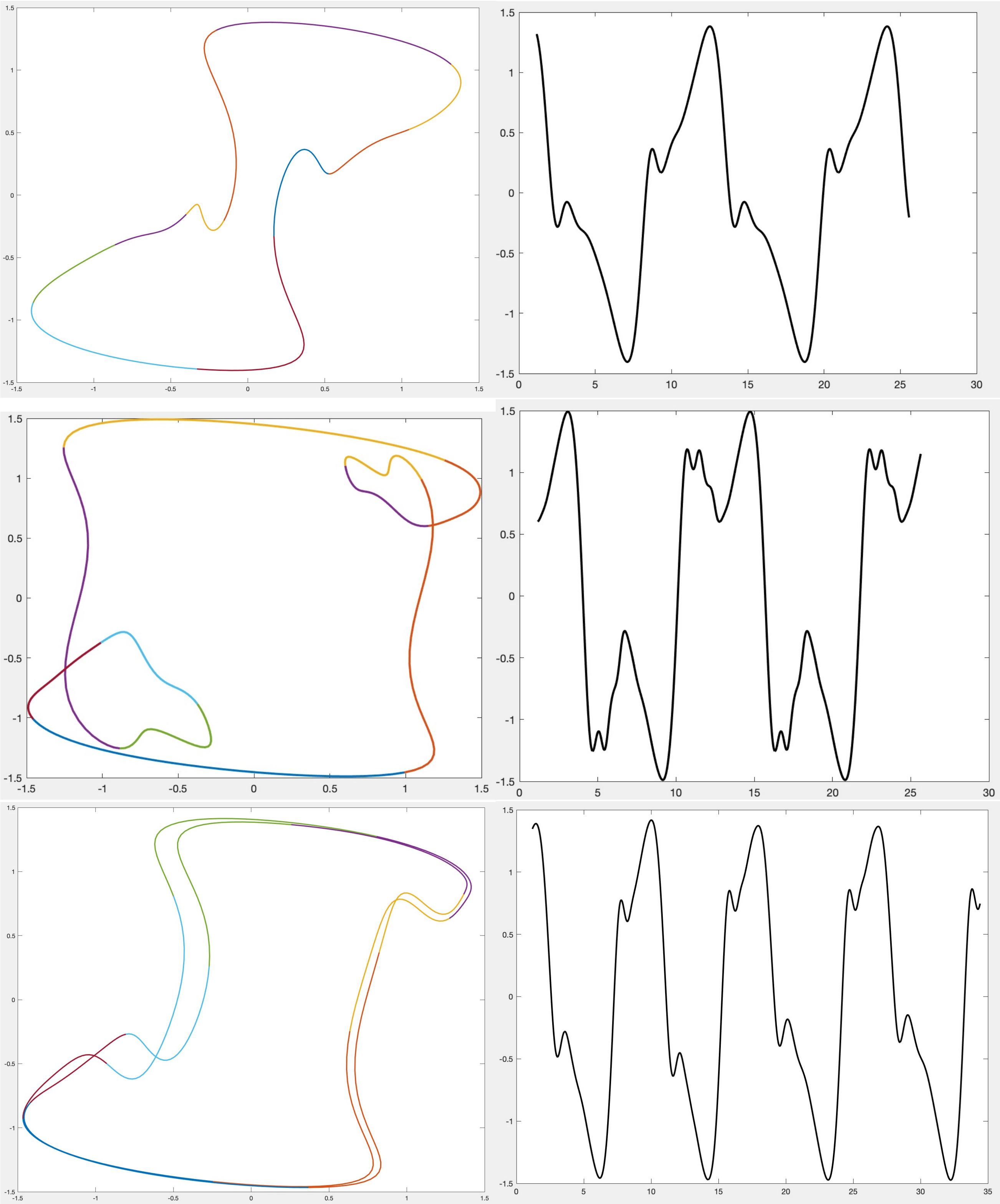

Fig. 3. Top left and right: a periodic orbit for $\varepsilon = 0.018$ and $\tau = 1.3875$, with period $T \sim 11.62$. Middle left and right: a periodic orbit for $\varepsilon = 0.0095$ and $\tau = 1.68607$, with period $T \sim 11.65$. Bottom left and right: a periodic orbit for $\varepsilon = 0.029$ and $\tau = 1.53$, with period $T = 16.61$

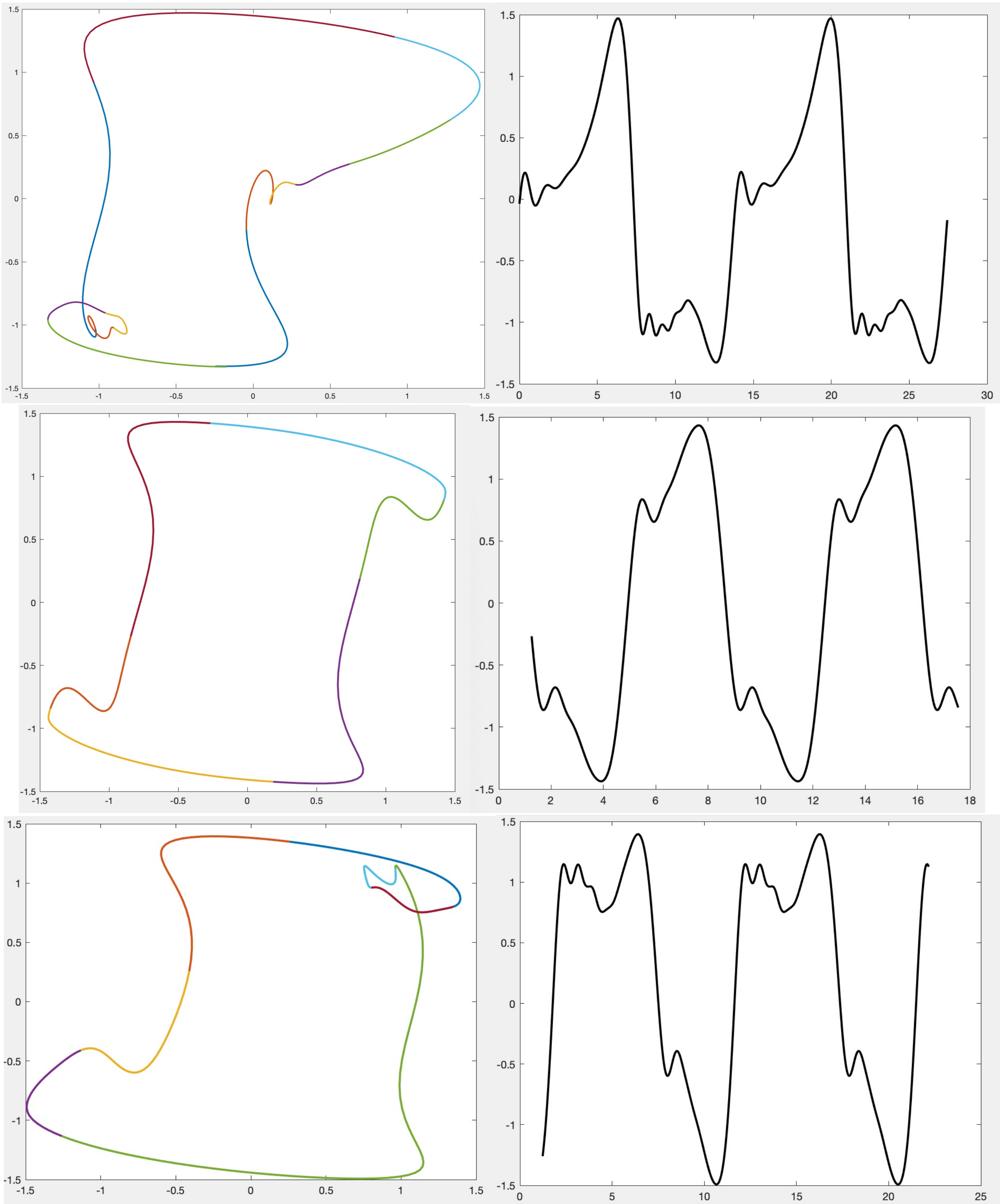

Fig. 4. Three coexisting periodic orbits when $\varepsilon = 0.02$ and $\tau = 1.65$. Top left and right: $T \sim 13.71$. Middle left and right: $T \sim 7.52$. Bottom left and right $T \sim 9.64$.

4. Final remarks and discussion

In the present work, the DDE under consideration admits a dependence upon the instantaneous state variable $x(t)$, which is only located in the expression defining the delay, and this dependence is moderated by the small parameter ε. This assumption is indeed crucial to deduce that the Picard Operator $\mathbf{L}_{x_0}$ and the Reduced Picard Operator $\Psi_{x_0,q}$ are contractions. The next step consists in generalizing the approach developed in this article to the case where the state dependent DDE also depends upon the undelayed state variable i.e., to study in a more general context, equations of the form

$$\dot{x} = \tau F(x(t), x(t - \delta(x(t))))$$

where $\delta(x(t))$ represents the state-dependent delay. In this more general context, we do not expect the Picard Operator nor the Reduced Picard Operator, at least in the way it is formulated in the present work, to be a contraction any longer. However, if we assume that the solutions of the above equation exist, this map still admits a unique fixed point. Therefore, to retrieve each solution, our approach will be to use yet again a Newton-Kantorovich argument to find the fixed point of the operator defined in (29). More precisely, the idea is to construct the operator

$$\mathbf{H} : \mathbb{P}_{q-1,b}[t] \mapsto \mathbb{P}_{q-1,b}[t], \quad z \mapsto \mathbf{H}(z) = z - \mathbf{B}\Big(z - \Psi_{x_0}(z) \Big), \tag{60}$$

where $\mathbf{B}$ is a linear operator close to $(\mathrm{Id} - D\Psi_{x_0}(z))^{-1}$. The main issue one could (and probably would) encounter by iterating the operator $\mathbf{H}$ directly is that its convergence is typically only guaranteed near the ensuing solution, which is, at this stage, unknown. One way to overcome this difficulty is to estimate the solution using any numerical integrator (see [Bellen & Zennaro, 2003; Bogacki & Shampine, 1989; Engelborghs *et al.*, 2002; Groothedde & Mireles-James, 2017; Khasi *et al.*, 2014; Krauskopf *et al.*, 2005; Lenz *et al.*, 2014; Sedaghat *et al.*, 2012; Torelli, 1989]). This estimated solution will then be the 'initial guess' we are looking for. Another way to construct a good 'initial guess' for the above operator is to construct the same procedure as in this article, but replacing the operator Ψ_{x_0} in (29) by

$$\Phi_{x_0,n}(z)(t) = z(0) + \int_0^{t/n} F(x(s), x(s - \delta(x(s))))ds, \quad 0 \leq t \leq \alpha_0, \tag{61}$$

where $n \geq 1$. In other words, instead of computing the Step Map directly on an interval of size $[0, \alpha_0]$, it is first to be computed on $[0, \alpha_0/n]$, and then on

$$[\alpha_0/n, 2\alpha_0/n], \dots, [(n-1)\alpha_0/n, \alpha_0].$$

We claim that for n sufficiently large, this operator is a contraction. This implies that the solution of the corresponding equation on the interval $[0, \alpha]$ is piecewise polynomial. We finally interpolate this latter (piecewise polynomial) solution on the interval $[0, \alpha_0]$ to get our initial guess for the operator defined in (60).

Even though our focus in the present work concerns the Cubic Ikeda Map, we claim that the techniques developed in this work are valid when the state variable belongs to a higher dimensional space. Indeed, we plan to implement these techniques, for instance and not limited to, the case of the following (delayed) Van der Pol system:

$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2(t - \delta(x)))y - x, \end{cases}$$

where μ is a parameter and $\delta(x) > 0$ represents the (state-dependent) delay and may take the form proposed in (1).

Finally, our future goal will include adapting techniques to be used for computer-assisted proofs. For instance, we need to find sufficient criteria to show that the Newton-like operator introduced in Section 2.4 is indeed a contraction nearby the ensuing fixed point and to estimate the size of a disk containing this fixed point on which the operator is a contraction.

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