# Periodic Orbits of a State-Dependent Delayed Van der Pol Equation

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## Abstract

We present a technique to find periodic orbits for the Van der Pol system with state-dependent delay. The approach presented involves computing fixed points of an appropriately defined Picard Operator and isolating periodic orbits by employing a multiple-shooting method. These tools are then rephrased in a computational context, utilizing the Lagrange-Chebyshev interpolation operator and Newton-like methods to numerically compute periodic orbits of this system.

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# 1 Introduction

The principle objective that lies at the heart of dynamical systems theory, whether it be studying a map, a flow, or a system of partial differential equations, is to understand precisely how a given system evolves over time. Perhaps the simplest types of dynamical systems are the linear systems of ordinary differential equations (ODEs), whose dynamics are completely characterized by exponential functions. However, even a small perturbation of a linear system of ODEs can give rise to complicated, chaotic dynamics. In a similar vain, the dynamics of delay differential equations (DDEs) tend to be far more complicated than their ODE counterparts. Due to their rich dynamics and potential for novel applications, the dynamical study of DDEs has received significant interest in recent decades [2, 8, 9, 10, 12, 14, 15, 17, 19, 20, 25, 26].

The underlying cause of a DDE's complexity lies in the mechanism that drives the system's evolution. For an ODE, the system evolves infinitesimally at time $t$ in response, either implicitly or explicitly, to the system's current state $x(t)$, the time $t$ itself, and certain immutable parameters. The evolution of a system of DDEs, however, depends also on certain *past* states of the system. In particular, given positive real numbers $\tau_1, \ldots, \tau_n$ for some $n \in \mathbb{N}$, the DDE will respond to the past states $x(t-\tau_1), \ldots, x(t-\tau_n)$ [7]. An immediate consequence of this difference in ordinary and delay differential equations is the dimension of their respective spaces of initial conditions. For an ODE system defined on an open set $U \subset \mathbb{R}^d$, an initial condition is simply a vector in $U$, and thus the space of initial conditions is finite-dimensional. However, for a DDE system whose output is in $\mathbb{R}^d$, an initial condition is a function (sometimes referred to as the initial *history* or initial *past* function) defined on the interval $(-\tau, 0)$, where $\tau = \max\{\tau_1, \ldots, \tau_n\}$. The space of initial conditions for a DDE is thus a function space, and necessarily infinite-dimensional.

In this work, we will not be studying delay differential equations as they were presented above, but instead their state-dependent counterparts. In other words, instead of the system responding to the fixed-delay state $x(t-\tau)$, it will instead depend on the state-dependent delay state $x\left(t-\tau+\varepsilon x(t)\right)$, for $|\varepsilon|$ sufficiently small. Of particular interest will be the state-dependent delayed Van der Pol equation

$$\ddot{x} - \mu(1 - x^2(t - \tau(1 - \varepsilon x(t))))\dot{x} + x = 0, \tag{1}$$

which can be equivalently characterized as

$$\left\{ \begin{array}{lcl} \dot{x} & = & y \\ \dot{y} & = & \mu(1 - x^2(t - \tau(1 - \varepsilon x(t))))y - x, \end{array} \right. \tag{2}$$

where $\mu$ and $\varepsilon$ are given parameters and $\tau > 0$ is the *centralized* (or *unperturbed*) delay. The classic Van der Pol [28] equation,

$$\ddot{x} - \mu(1 - x^2)\dot{x} + x = 0, \tag{3}$$

is often viewed as the precursor for the development of nonlinear oscillation theory. Here, $x(t)$ represents the amplitude of an oscillating current and $\mu$ is the damping parameter, controlling the effect of the nonlinear term. This equation has been used as a model for certain electronic systems, the dust density in dusty plasmas, as well as earthquake faults [1, 14, 27, 29], and thus has garnered interest from researchers in electronics, engineering, physics, and mathematics. The Van der Pol equation with delayed feedback has also been studied, as can be seen in [3, 9]. In fact, according to [3], indicators of chaos within the delayed Van der Pol equation have been observed for certain parameter choices. As will be seen in Section 5.3, evidence of chaos is also observed in the state-dependent delay Van der Pol equation for certain parameters. A thorough study of chaos and bifurcations within this system is beyond the scope of this work, but would make for an interesting future pursuit.

Due to their complexity, the methods used to study DDEs differ from those used for ODEs. Even when the delay term is constant, one cannot directly use the classical ODE methods to solve a DDE. For example, consider the equation

$$\begin{cases} \dot{x}(t) = F\left[x(t-\tau)\right], & t \geq 0, \\ x(t) = x_0(t), & -\tau \leq t < 0, \end{cases} \tag{4}$$

where $F$ is an integrable function. One may notice that the solution to (4) on the interval $[0, \tau)$ depends entirely on the history function $x_0(t)$, whereas the solution on the interval $[\tau, 2\tau)$ depends on the one found on $[0, \tau)$. Thus, if one wants to find a global solution to (4), then it must be done iteratively on intervals of length $\tau$. This is sometimes referred to as the *method of steps* [21, 22, 29] and is often seen as the most basic approach to solving a DDE.

When the delay term is state-dependent, however, the situation becomes far more nuanced. Consider now the equation

$$\begin{cases} \dot{x}(t) = F\left[x(t-\tau(1-\epsilon x(t)))\right], & t \geq 0, \\ x(t) = x_0(t), & t < 0, \end{cases} \tag{5}$$

where $F$ is an integrable function. If one were to try using the method of steps to calculate a solution of (5), then the step sizes need to be chosen more carefully than when solving (4). In particular, one might like to find $c \in \mathbb{R}^+$ such that the solution $x(t)$ on the interval $[0, c)$ only depends on the given history $x_0(t)$. If $c$ is chosen too large, however, then this will not be possible. On the other hand, an arbitrarily small choice of $c$ would work in theory, but computing a long-time solution $x(t)$ may then become infeasible. One possible remedy for this interval-size dilemma, as explored in [4] in the case of the delayed Cubic Ikeda equation, is to consider a Picard-like operator which first computes a solution on a smaller interval, but then "retraces" its steps to find a solution on a larger interval of interest.

This last method works particularly well when searching for periodic solutions of (5) as it allows us to compute the solution on intervals whose sizes are not too small, but evenly divide the period. This method differs from the similar Picard-like method introduced in [22] in that, by integrating over a larger time-interval, the computation of periodic orbits becomes more easily obtainable. In this paper, we further explore the computation of periodic orbits in state-dependent DDEs, but in the case of the Van der Pol system outlined in (2). As will be seen in the following sections, the methods used to solve (2) differ from those used to solve (5), as the Van der Pol system depends both on a delayed and undelayed state. It is important to disclose that, while we do indeed compute and illustrate many periodic orbits for the system (2), the proposed method cannot guarantee us to be able to calculate all periodic orbits for given parameters $\mu, \tau$, and $\epsilon$. For results which prove the existence and persistence of periodic orbits in a separate class of state-dependent delay differential equations, see [13].

## 2 Preliminaries

When discussing delay differential equations, one must be careful with how to properly define a solution. The most general setting to study Eq. (2) must take into consideration the solution's 'history' function as an initial condition. We therefore consider the following equation:

$$\begin{cases} \dot{x} = y, \qquad \dot{y} = \mu(1 - x^2(t - \tau\,(1 - \varepsilon x(t))))y - x, & t > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad x(t) = \varphi(t) & t < 0, \end{cases}$$

where $\varphi$ is the (given) initial history function for $x$ on the interval $(-\infty, 0)$. Also recall that, in this system, $\mu$ is the damping parameter and $\varepsilon$ is a small parameter that perturbs the initial delay $\tau$. By using a time rescaling of the form $t = \tau u$, and setting

$$\tilde{x}(u) = x(\tau u), \quad \tilde{y}(u) = y(\tau u) \;\text{ and }\; \tilde{\varphi}(u) = \varphi(\tau u),$$

our original system is equivalent to the following, more convenient system:

$$\begin{cases} \tilde{x}' = \tau\tilde{y}, \qquad \tilde{y}' = \tau\mu(1 - \tilde{x}^2(u - 1 + \varepsilon\tilde{x}(u)))\tilde{y} - \tau\tilde{x}, & u > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \tilde{x}(u) = \tilde{\varphi}(u) & u < 0, \end{cases}$$

where $'$ denotes the derivative with respect to $u$. For convenience, however, we would like to use the variable $t$ instead of $u$, use "$\cdot$" instead of "$'$", and remove the hats and tildes in the above system. This gives rise to the more digestible system below, to which we will be referring to throughout the rest of this article:

$$\begin{cases} \dot{x} = \tau y, \qquad \dot{y} = \tau\mu\left(1 - x^2\left(t - 1 + \varepsilon x(t)\right)\right)y - \tau x, & t > 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad x(t) = \varphi(t) & t < 0. \end{cases} \tag{6}$$

At times, we will sometimes adopt the following more compact but equivalent notation for equation (6)

$$\dot{\mathbf{z}} = \mathbf{H}_{\varphi,\mathbf{z}}(t) \coloneqq \tau\mathbf{F}(\mathbf{z}(t), x(t - 1 + \varepsilon x(t))), \quad t > 0, \tag{7}$$

where $\mathbf{z}(t) \coloneqq \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and

$$\mathbf{F}(\mathbf{z}, u) = \begin{pmatrix} y \\ \mu(1-u^2)y - x \end{pmatrix},$$

and $x(t) = \varphi(t)$ for $t < 0$. Now that the DDE has been properly defined, we will now demonstrate how to estimate the solution of Eq. (6) given an initial history function.

# 3 Step Maps

In what follows, we elect to use the following norm on $\mathbb{R}^2$:

$$\left\| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right\| = \max\{|u_1|, |u_2|\}.$$

Let $\mathrm{I} \subset \mathbb{R}$ be an interval and $b > 0$. We say that $g \in \mathbf{C}^1_b(\mathrm{I})$ where

$$g : \mathrm{I} \to \mathbb{R},\ t \mapsto g(t),$$

if $g$ is continuous and piecewise $\mathbf{C}^1$ on I and $|g'(t)| \leq b$ whenever $g'(t)$ is well-defined. (Note: when we say $g$ is piecewise $\mathbf{C}^1$ on I, we mean that $g$ is differentiable everywhere except for a finite number of points in the interior of I). Now, let $\alpha > 1$, $b_0 > 0$, and $\varphi \in \mathbf{C}^1_{b_0}([-\alpha, 0])$ In what follows, we will choose $b > b_0$ and $\mathrm{M} > 1$ large enough such that

$$\sup_{t \in [-\alpha, 0]} \left\| \begin{pmatrix} \varphi(t) \\ \dot{\varphi}(t)/\tau \end{pmatrix} \right\| \leq \mathrm{M}/2 \tag{8}$$

and

$$\tau \sup_{|\mathbf{z}| \leq \mathrm{M}, |u| \leq \mathrm{M}} \|\mathbf{F}(\mathbf{z}, u)\| + 1 \leq b. \tag{9}$$

Let $\beta > 0$. We define

$$\mathbf{S}_{\varphi,\beta} = \{\mathbf{f} = (f_1, f_2) \in \mathbf{C}^1_b([0,\beta]) \times \mathbf{C}^1_b([0,\beta]) :$$

$$\|\mathbf{f}(t)\| \leq \mathrm{M},\ 0 \leq t \leq \beta,\ f_1(0) = \varphi(0),\ f_2(0) = \varphi'(0)/\tau\}.$$

The set $\mathbf{S}_{\varphi,\beta}$ is equipped with the metric

$$\mathbf{d}_\beta(\mathbf{f}, \mathbf{g}) \coloneqq \|\mathbf{f} - \mathbf{g}\|_\beta \coloneqq \max_{j=1,2}\{ \sup_{t \in [0,\beta]} |f_j(t) - g_j(t)| \}.$$

(The authors note here that $\mathbf{S}_{\varphi,\beta}$ is not a vector space and thus the notation $\|\mathbf{f} - \mathbf{g}\|_\beta$ will only be used for convenience and when it makes sense to do so.)

### 3.1 Picard Operators

The goal of this section is to estimate the solution of (6) on the interval $[0,\alpha]$. We will shortly see that such a solution is the fixed point of a specific Picard operator. To show that this operator admits a unique fixed point, we will show that the same Picard Operator restricted on a sufficiently small interval is a contraction. We then deduce that this operator admits a unique fixed point when the set of functions are defined on $[0,\alpha]$.

Let $0<\beta\le\alpha$ and $\varphi\in C_b^1([-\alpha,0])$. We first define

$$\Psi_{\varphi,\beta} : \mathbf{S}_{\varphi,\beta} \to \mathbf{S}_{\varphi,\beta}$$

$$\mathbf{z} := (x,y) \mapsto \Psi_{\varphi,\beta}(\mathbf{z})$$

where

$$\Psi_{\varphi,\beta}(\mathbf{z})(t) := \begin{pmatrix} \varphi(0) \\ \dot{\varphi}(0)/\tau \end{pmatrix} + \tau \int_0^t \mathbf{F}(\mathbf{z}(s), \tilde{x}(s-1+\varepsilon x(s))ds \tag{10}$$

where we define

$$\tilde{x}(u) := \begin{cases} x(u), & \text{if} \quad u>0, \\ \varphi(u), & \text{if} \quad u\le 0. \end{cases}$$

Equivalently, we can write $\Psi_{\varphi,\beta}(\mathbf{z})(t) = \Big(\Psi^x_{\varphi,\beta}(\mathbf{z})(t), \Psi^y_{\varphi,\beta}(\mathbf{z})(t)\Big)$, where

$$\begin{aligned} \Psi^x_{\varphi,\beta}(\mathbf{z})(t) &= \varphi(0) + \tau\int_0^t y(s)ds \\ \Psi^y_{\varphi,\beta}(\mathbf{z})(t) &= \frac{\dot{\varphi}(0)}{\tau} + \tau\int_0^s \Big(\mu\Big[1-\tilde{x}^2(s-1+\varepsilon x(s))\Big]y(s) - x(s)\Big)ds. \end{aligned}$$

This construction naturally leads us to the following theorem.

**Theorem 1** *There exists $0<\beta_0\le\alpha$ such that, for all $0<\beta\le\beta_0$, $\Psi_{\varphi,\beta}$ is a contraction. More precisely, we have*

$$\mathbf{d}_\beta\Big(\Psi_{\varphi,\beta}(\mathbf{f}), \Psi_{\varphi,\beta}(\mathbf{g})\Big) \le \frac{1}{2}\mathbf{d}_\beta(\mathbf{f},\mathbf{g}).$$

The proof of Theorem 1 is rather technical and is given in the appendix, though a more general proof of this theorem for state-dependent delay differential equations with Lipschitz delay term can be found in [36, 37].

Since our search is for periodic orbits of Eq. (6), it is reasonable to assume that every component of the solution is bounded. More precisely, without lost of generality we assume that the upper bound stated in (8) always remains valid, i.e. for each solution $\mathbf{z}(t) = (x(t), y(t))$, we have

$$\left\| m \begin{pmatrix} x(t) \\ \dot{x}(t)/\tau \end{pmatrix} \right\| \leq \mathrm{M}/2, \qquad \forall t > 0. \tag{11}$$

Under the above assumption, the former proposition admits the following corollary.

**Corollary 1** *For all $1 < \alpha < 2$ and for all $\varphi \in C_b^1([-\alpha, 0])$, the operator $\Psi_{\varphi,\alpha}$ admits a unique fixed point, i.e. there exists $\mathbf{z}_\varphi \in \mathbf{S}_{\varphi,\alpha}$ such that*

$$\Psi_{\varphi,\alpha}(\mathbf{z}_\varphi) = \mathbf{z}_\varphi,$$

*and therefore $z_\varphi$ is the unique solution of (6) on $[0, \alpha]$.*

Proof of Corollary 1: For clarity, we rename our initial history function on $[-\alpha, 0]$:

$$\varphi_0(t) \coloneqq \varphi(t), \quad -\alpha \leq t < 0.$$

Observe that since the first line of Eq. (6) writes $\dot{x} = \tau y$, we have that $\mathbf{z}(t)$ is uniquely defined by $x(t)$, as second component can be deduced by writing $y(t) = \dot{x}(t)/\tau$. Thanks to Theorem 1, Eq. (6) admits a unique solution $\mathbf{z}_{\varphi_0}(t) = (x_{\varphi_0}(t), y_{\varphi_0}(t))$ on the interval $[0, \beta]$. We then write

$$\varphi_1(t) = x_{\varphi_0}(t + \beta), \text{ if } -\beta \leq t \leq 0, \quad \varphi_1(t) = \varphi_0(t + \beta), \text{ if } t \leq -\beta.$$

We see that $\varphi_1 \in \mathbf{C}_b^1([-\alpha, 0])$ and satisfies (11). We may then deduce the solution of (6) on the interval $[\beta, 2\beta]$ by considering the fixed point of $\Psi_{\varphi_1,\beta}$, i.e., we write

$$\mathbf{z}_{\varphi_1} = (x_{\varphi_1}, y_{\varphi_1}) = \Psi_{\varphi_1,\beta}(x_{\varphi_1}, y_{\varphi_1}).$$

Again, we write

$$\varphi_2(t) = x_{\varphi_1}(t + \beta), \text{ if } -\beta \leq t \leq 0, \quad \varphi_2(t) = \varphi_1(t + \beta), \text{ if } t \leq \beta.$$

Now, assume we have defined

$$\mathbf{z}_{\varphi_0}, \mathbf{z}_{\varphi_1}, \ldots, \mathbf{z}_{\varphi_{k-1}} \quad \text{and} \quad \varphi_1, \varphi_2, \ldots, \varphi_k,$$

for some $k > 2$. Observe that all the $\varphi_j$'s belong to $\mathbf{C}_b^1([-\alpha, 0])$ and satisfy (11). We may then deduce the solution of (6) on the interval $[(k\beta, (k+1)\beta]$ by considering the fixed point of $\Psi_{\varphi_k,\beta}$, i.e., we write

$$\mathbf{z}_{\varphi_k} = (x_{\varphi_k}, y_{\varphi_k}) = \Psi_{\varphi_k,\beta}(\mathbf{z}_{\varphi_k}),$$

and the $x$-component of the solution of (6) on $[0, (k+1)\beta]$ writes

$$x(t) = x_{\varphi_{j-1}}(t - (j-1)\beta) = \varphi_j(t - j\beta), \text{ if } (j-1)\beta \leq t < j\beta, \quad 1 \leq j \leq k+1.$$

Thus, Eq. (6) admits a unique solution $\mathbf{z}_\varphi$ on $[0, \alpha]$.

### 3.2 Lagrange-Chebyshev interpolation

The main difficulty that one encounters when dealing with the above Picard operator is that its range is of infinite dimension. Since we are interested in explicitly computing periodic solutions to (6), working in an infinite-dimensional space will be infeasible. To overcome this issue, we aim to restrict the domain and range of the operator to a finite-dimensional space. This will allow us to introduce a new operator, called the *Reduced Picard Operator* (see next subsection), of which fixed points will approximate fixed points of the former, infinite-dimensional operator. We succeed in doing this by using Lagrange-Chebyshev interpolation.

Let $\alpha_0 > 1$, $q > 1$ be an integer, and $\mathbb{P}_q[t]$ be the space of polynomial functions in the variable $t$ of degree less or equal than $q-1$. We first define

$$\mathrm{R} : [0, \alpha_0] \to [-1, 1], \quad t \mapsto -1 + \frac{2}{\alpha_0} t,$$

a linear map that rescales the interval $[0, \alpha_0]$ to the interval $[-1, 1]$ in a one-to-one manner, preserving the orientation. We then define the interpolation operator

$$\mathcal{L}_q : \mathbf{C}_b^1([0, \alpha_0]) \to \mathbb{P}_q, \ \mathbf{g} \mapsto \mathcal{L}_q(\mathbf{g})$$

by

$$\mathcal{L}_q(\mathbf{g})(t) = \sum_{j=0}^{q-1} c_j \left(T_j \circ \mathrm{R}\right)(t),$$

where the $T_j$'s ($j = 0, \ldots, q-1$) are the Chebyshev polynomials of the first kind,

$$T_j(t) = \cos\left(j \arccos(t)\right), \ j = 0, \ldots, q-1,$$

and for $j = 0, \ldots, q-1$,

$$c_j = \frac{2}{q} \sum_{k=0}^{q-1} \left(\mathbf{g} \circ \mathrm{R}^{-1}\right)(u_k) T_j(u_k), \quad j > 0, \qquad \text{and} \qquad c_0 = \frac{1}{q} \sum_{k=0}^{q-1} \left(\mathbf{g} \circ \mathrm{R}^{-1}\right)(u_k),$$

where the $u_k$'s are the Chebyshev nodes on $[-1, 1]$, i.e.,

$$u_k = \cos\left(\frac{2k+1}{2q}\, \pi\right), \quad k = 0, \ldots, q-1.$$

Similarly, we define the same interpolation operator, but this time on the interval $[1, \alpha_0]$,

$$\tilde{\mathcal{L}}_q : \mathbf{C}_b^1([1, \alpha_0]) \to \mathbb{P}_q, \ \mathbf{g} \mapsto \tilde{\mathcal{L}}_q(\mathbf{g})$$

where

$$\tilde{\mathcal{L}}_q(\mathbf{g})(t) = \sum_{j=0}^{q-1} \tilde{c}_j \left(T_j \circ \tilde{\mathrm{R}}\right)(t)$$

where

$$\tilde{\mathrm{R}} : [1, \alpha_0] \to [-1, 1],\ t \mapsto -1 + \frac{2}{\alpha_0 - 1}(t-1).$$

and for $j = 0, \ldots, q-1$,

$$\tilde{c}_j = \frac{2}{q}\sum_{k=0}^{q-1} \left(\mathbf{g} \circ \tilde{\mathrm{R}}^{-1}\right)(u_k) T_j(u_k), \quad j > 0, \qquad \text{and} \qquad \tilde{c}_0 = \frac{1}{q}\sum_{k=0}^{q-1} \left(\mathbf{g} \circ \tilde{\mathrm{R}}^{-1}\right)(u_k).$$

See [38] for more details.

We now extend the above definitions to vector-valued functions. More precisely, if we write

$$\mathbf{f} = \begin{pmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \end{pmatrix},$$

then we have

$$\mathcal{L}_q(\mathbf{f}) := \begin{pmatrix} \mathcal{L}_q(\mathrm{f}_1) \\ \mathcal{L}_q(\mathrm{f}_2) \end{pmatrix}, \quad \tilde{\mathcal{L}}_q(\mathbf{f}) := \begin{pmatrix} \tilde{\mathcal{L}}_q(\mathrm{f}_1) \\ \tilde{\mathcal{L}}_q(\mathrm{f}_2) \end{pmatrix}.$$

These operators are linear projections and the following lemma holds.

**Lemma 1** *Let $b > 1$ and let $\mathrm{f}_1, \mathrm{f}_2 \in \mathbf{C}^1_b([0, \alpha_0])$. Then for i=1,2,*

$$\sup_{0 \le t \le \alpha_0} |\mathcal{L}_q(\mathrm{f}_i)(t) - \mathrm{f}_i(t)| \le \frac{\alpha_0(1 + \mu_q)}{2q}\, b,$$

*where*

$$\mu_q = \frac{1}{\pi}\sum_{j=0}^{q-1} \cot\left(\frac{(j+1/2)\pi}{2q}\right) \sim \frac{2}{\pi}\log(q) + 0.9625 + \mathcal{O}(1/q).$$

*Hence, by writing $\mathbf{f} = \begin{pmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \end{pmatrix}$, we have*

$$\mathbf{d}_{\alpha_0}(\mathcal{L}_q(\mathbf{f}), \mathbf{f}) \le \frac{\alpha_0(1 + \mu_q)}{2q} b.$$

*Similarly if $\tilde{f} \in \mathbf{C}^1_b([1, \alpha_0])$, then*

$$\sup_{1 \le t \le \alpha_0} |\tilde{\mathcal{L}}_q(\tilde{f})(t) - \tilde{f}(t)| \le \frac{(\alpha_0 - 1)(1 + \mu_q)}{2q} b.$$

This lemma is a direct consequence of Jackson's Theorem and its Corollary 6.14A in [38].

## 3.3 The Reduced Picard Operator

Let $1 < \alpha < \alpha_0$, $\varphi \in C^1_b([-\alpha, 0])$, and $q > 1$ be an integer. We define the **Reduced Picard Operator**

$$\hat{\Psi}_{\varphi,q} : \mathbb{P}_q \times \mathbb{P}_q \to \mathbb{P}_q \times \mathbb{P}_q, \quad \mathbf{z} = (x, y) \mapsto \hat{\Psi}_{\varphi,q}(\mathbf{z}) = \begin{pmatrix} \hat{\Psi}_{\varphi,q,x}(\mathbf{z}) \\ \hat{\Psi}_{\varphi,q,y}(\mathbf{z}) \end{pmatrix}$$

by

$$\hat{\Psi}_{\varphi,q}(\mathbf{z})(t) = \begin{pmatrix} \varphi(0) \\ \dot{\varphi}(0)/\tau \end{pmatrix} + \tau \int_0^t \mathcal{L}_{q-1}\Big(\mathbf{F}(\mathbf{z}(s), \tilde{x}(s - 1 + \varepsilon x(s))\Big)\, ds,$$

or, equivalently,

$$\begin{aligned} \hat{\Psi}_{\varphi,q,x}(\mathbf{z}) &= \varphi(0) + \tau \int_0^t \mathcal{L}_{q-1}(y(s))\, ds \\ \hat{\Psi}_{\varphi,q,y}(\mathbf{z}) &= \frac{\dot{\varphi}(0)}{\tau} + \tau \int_0^t \mathcal{L}_{q-1}\Big(\mu\left(1 - \tilde{x}^2\left(s - 1 + \varepsilon x(s)\right)\right) y(s) - x(s), \Big)\, ds \end{aligned} \tag{12}$$

recalling that

$$\tilde{x}(u) = x(u) \text{ if } u > 0, \text{ and } \tilde{x}(u) = \varphi(u) \text{ if } u \le 0.$$

Note that a fixed point of the above operator is a solution (on the interval $[0, \alpha_0]$) of the system

$$\begin{cases} \dot{x} = \tau \mathcal{L}_{q-1}(y), \\ \dot{y} = \tau \mathcal{L}_{q-1}\Big(\mu\Big(1 - \tilde{x}^2[t - 1 + \varepsilon x(t)] y(t)\Big) - x(t)\Big), \end{cases} \tag{13}$$

with

$$x(t) = \varphi(t), \quad -\alpha \le t \le 0.$$

We shall verify *a posteriori* that the solution of the above equation can be arbitrarily close to the solution of (6), see last section for more details. This is demonstrated in the following theorem.

**Theorem 2** *Let $\delta > 0$ and $1 < \alpha < \alpha_0$. For all $\varphi \in C^1_b([-\alpha, 0])$, there exists $q_1 > 1$ such that, for all $q \ge q_1$,*

$$\mathbf{d}_\alpha(\hat{\mathbf{z}}_\varphi, \mathbf{z}_\varphi) \le \delta,$$

*where*

$$\hat{\Psi}_{\varphi,q}(\hat{\mathbf{z}}_\varphi) = \hat{\mathbf{z}}_\varphi, \quad \text{and} \quad \Psi_{\varphi,\alpha}(\mathbf{z}_\varphi) = \mathbf{z}_\varphi.$$

The proof of this theorem is rather technical and can be found in the appendix.

As a consequence, for a given past $\varphi \in C^1_b([-\alpha, 0])$, the fixed point of the Picard Operator, and therefore the solution of (6) on the interval $[0, \alpha_0]$ can be computed with any desired precision by computing the fixed point of the Reduced Picard Operator as long as we choose $q$, i.e. the number of nodes for the interpolation, sufficiently large.

### 3.4 A first Newton-like operator

For a given history function $\varphi$, to compute the fixed point of $\hat{\mathbf{\Psi}}_{\varphi,q}$, we introduce the Newton-like operator

$$\mathcal{N}_{\varphi,q} : \mathbb{P}_q \to \mathbb{P}_q,\ \mathbf{z} \mapsto \mathbf{z} - \hat{\mathbf{L}} \circ \left( \hat{\Psi}_{\varphi,q}(\mathbf{z}) - \mathbf{z} \right),$$

where $\hat{\mathbf{L}}$ is an operator "close" to $(d\hat{\Psi}_{\varphi,q}(\mathbf{z}) - \mathrm{Id})^{-1}$, Id being the identity operator. More precisely, we choose

$$\hat{\mathbf{L}} = \left( (d\hat{\mathbf{\Psi}}_{\varphi,q} - \mathrm{Id})\Big|_{\varepsilon=0} \right)^{-1}.$$

To compute the Reduced Picard Operator and the Newton-like operator above, we must first identify each polynomial with its coefficients. To do this, we introduce the following isomorphisms:

$$\mathbf{J} : \mathbb{R}^q \to \mathbb{P}_q,\ \mathbf{a} = (a_0, \ldots, a_{q-1}) \mapsto \sum_{j=0}^{q-1} a_j t^j,$$

$$\mathcal{J} : \mathbb{R}^q \times \mathbb{R}^q \to \mathbb{P}_q \times \mathbb{P}_q,\ (\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{J}(\mathbf{a}), \mathbf{J}(\mathbf{b})).$$

If we write

$$\mathbf{z}(t) = (x(t), y(t)), \quad x(t) = \sum_{j=1}^{q} x_{j-1} t^{i-1}, \quad y(t) = \sum_{j=1}^{q} y_{j-1} t^{i-1},$$

then we may also write

$$d\hat{\mathbf{\Psi}}_{\varphi,q} = \mathcal{J} \circ \mathbf{L}_{\varphi,q} \circ \mathcal{J}^{-1}.$$

To compute this operator $\mathbf{L}_{\varphi,q}$, we express it in the form

$$\mathbf{L}_{\varphi,q}(\mathbf{z}) = \begin{pmatrix} \mathbf{L}_1(\mathbf{z}) & \mathbf{L}_2(\mathbf{z}) \\ \mathbf{L}_3(\mathbf{z}) & \mathbf{L}_4(\mathbf{z}) \end{pmatrix}.$$

It is easy to verify that $\mathbf{L}_1 = 0$, (i.e., the $q \times q$ matrix with 0 entries). Furthermore, for each $\ell = 2, 3, 4$ when writing

$$\mathbf{L}_\ell = \left( L^\ell_{i,j} \right)_{1 \le i \le q,\ 1 \le j \le q}$$

we have

$$L^2_{i,j} = \tau/j \text{ if } i = j+1,\ j \le q-1, \qquad L^2_{i,j} = 0 \text{ if } i \ne j+1,\ j \le q-1,$$

and

$$L^2_{1,q} = 0, \text{ and } L^2_{i,q} = \tau c_{i-2}/(i-1), \quad 2 \le i \le q,$$

where

$$\mathcal{L}_{q-1}(t^{q-1}) = \sum_{i=1}^{q-1} c_{i-1} t^{i-1}.$$

By definition we have

$$\frac{\partial \hat{\mathbf{\Psi}}_{\varphi,q,y}(\mathbf{z})}{\partial x_{j-1}} = \sum_{i=1}^{q-1} L^3_{i,j} t^{i-1}, \qquad \frac{\partial \hat{\mathbf{\Psi}}_{\varphi,q,y}(\mathbf{z})}{\partial y_{j-1}} = \sum_{i=1}^{q-1} L^4_{i,j} t^{i-1}.$$

For each integer $j = 1, \dots, q$, we have

$$\begin{aligned} \frac{\partial \hat{\mathbf{\Psi}}_{\varphi,q,y}(\mathbf{z})}{\partial x_{j-1}} &= \tau \int_0^t \mathcal{L}_{q-1}\Big( -2\tau\mu \tilde{x}(s-1) y(s) \frac{\partial \tilde{x}}{\partial x_{j-1}}(s-1) - \frac{\partial \tilde{x}}{\partial x_{j-1}}(s-1) \Big) ds \\ \frac{\partial \hat{\mathbf{\Psi}}_{\varphi,q,y}(\mathbf{z})}{\partial y_{j-1}} &= \tau\mu \int_0^t \mathcal{L}_{q-1}\Big( (1 - \tilde{x}^2(s-1)) \frac{\partial y}{\partial y_{j-1}}(s) \Big) ds \end{aligned}$$

where

$$\tilde{x}(u) = x(u) \ \text{ if } u > 0, \quad \tilde{x}(u) = \varphi(u) \ \text{ if } u \le 0.$$

Therefore

$$\frac{\partial \tilde{x}}{\partial x_{j-1}}(t-1) = 0 \ \text{ if } t \le 1, \quad \frac{\partial \tilde{x}}{\partial x_{j-1}}(t-1) = (t-1)^{j-1} \text{ if } \ t > 1,$$

and

$$\frac{\partial y}{\partial y_{j-1}}(t) = t^{j-1}.$$

Numerical evidence suggests that this Newton-like operator is indeed a contraction, (for initial values close enough to the ensuing solution) however a formal proof of this can not be presented at this time. An implementation of this algorithm is presented in Section 5.

### 3.5 The Step Map and the multi-shooting map

We are now in the position to introduce the Step Map, which will allow us to compute periodic solutions to (6). Let $\alpha > 1$ as above and $q \ge 1$ be an integer. For a given $\varphi \in \mathbb{P}_q$, we denote by

$$\hat{\mathbf{z}}_\varphi(t) = (\hat{x}_\varphi(t), \hat{y}_\varphi(t)) \tag{14}$$

the fixed point of the Reduced Picard Operator defined above. We may now define the *Step Map*

$$\mathbf{S}_{\alpha,q} : \mathbb{P}_q \to \mathbb{P}_q, \quad \varphi \mapsto \mathbf{S}_{\alpha,q}(\varphi)$$

by

$$\mathbf{S}_{\alpha,q}(\varphi)(t) = \hat{x}_\varphi(t+\alpha).$$

Due to Theorem 2, a periodic solution of (6) is approximated by a periodic solution of (13) with the same period $\mathrm{T} > 0$. Thanks to the above definition, a solution of (13) is periodic if there exists $\alpha > 1$, an integer $p \geq 1$, and $\varphi \in \mathbb{P}_q$ such that

$$\mathbf{S}^p_{\alpha,q}(\varphi) \coloneqq \mathbf{S}_{\alpha,q} \circ \mathbf{S}^{p-1}_{\alpha,q}(\varphi) = \varphi.$$

This implies that $\mathrm{T} = p\alpha$ and, as T is *a-priori* unknown, calculating T amounts to finding $\alpha > 1$ and the integer $p$. The strategy to do so consists of fixing the integer $p$ and finding $1 < \alpha \leq \alpha_0$ using a different Newton-like operator (to be introduced shortly). In this context, and for each iteration of this operator, $\alpha_0$ will remain fixed. For the periodic orbits displayed at the end of Section 5, we set $\alpha_0 = 1.4$.

To account for the fact that shifted periodic solutions of (6) are themselves also periodic solutions (see [4] for more details and discussion), we need to impose the value of $\varphi$ at the origin. We set

$$\varphi(0) = \gamma,$$

where $\gamma$ is to be chosen in the interior of the range of the ensuing solution. *A-posteriori*, we will be able to choose $\gamma = 0$.

We define the following injection:

$$\mathbf{J}^* : \mathbb{R}^{q-1} \to \mathbb{P}_q, \ \mathbf{b} = (b_1, \dots, b_{q-1}) \mapsto \gamma + \sum_{j=1}^{q-1} b_j t^j.$$

Let $p$ be an integer and let $\mathrm{R}_q = [1, 3/2] \times \mathbb{R}^{q-1}$. Each element in $\mathrm{R}_q$ writes as $\mathbf{a}_1^* = (\alpha, \mathbf{a}_1)$ where

$$\mathbf{a}_1 = (a_{1,1}, a_{1,2}, \dots, a_{1,q-1}).$$

We may then define the following multi-shooting map:

$$\mathbf{G} : \mathrm{R}_q \times \underbrace{\mathbb{R}^q \times \cdots \times \mathbb{R}^q}_{p-1 \text{ times}} \to \underbrace{\mathbb{R}^q \times \cdots \times \mathbb{R}^q}_{p \text{ times}},$$

$$(\mathbf{a}_1^*, \mathbf{a}_2, \dots, \mathbf{a}_p) \mapsto \Big(\mathbf{J}^{-1}(\mathbf{S}_{\alpha,q}(\mathbf{J}^*(\mathbf{a}_1)) - \mathbf{J}(\mathbf{a}_2)), \mathbf{J}^{-1}(\mathbf{S}_{\alpha,q}(\mathbf{a}_2)) - \mathbf{J}(\mathbf{a}_3)) \dots,$$

$$\mathbf{J}^{-1}(\mathbf{J}(\mathbf{S}_{\alpha,q}(\mathbf{J}(\mathbf{a}_{p-1})) - \mathbf{J}(\mathbf{a}_p)), \mathbf{J}^{-1}(\mathbf{J}(\mathbf{S}_{\alpha,q}(\mathbf{J}(\mathbf{a}_p)) - \mathbf{J}(\gamma, \mathbf{a}_1))\Big)$$

where

$$(\gamma, \mathbf{a}_1) = (\gamma, a_{1,1}, a_{1,2}, \dots, a_{1,q-1}).$$

Observe that if

$$\mathbf{W} = (\mathbf{w}_1^*, \mathbf{w}_2, \dots, \mathbf{w}_{p-1})$$

satisfies $\mathbf{G}(\mathbf{W}) = 0$, then $\mathbf{J}^*(\mathbf{w}_1)$ represents a periodic solution of (13). More specifically, $\mathbf{J}^*(\mathbf{w}_1)$ denotes the first coordinate of the periodic solution, but, through the relation $y(t) = \dot{x}(t)/\tau$, we may uniquely determine the second coordinate as $\frac{1}{\tau} \cdot \frac{d}{dt}\mathbf{J}^*(\mathbf{w}_1)$ . Thus, our search is for zeros of the map $\mathbf{G}$, and so we introduce the following Newton-like operator:

$$\begin{aligned} \mathbf{H}: \qquad & \mathrm{R}_q \times \underbrace{\mathbb{R}^q \times \cdots \times \mathbb{R}^q}_{p-1 \text{ times}} \to \mathrm{R}_{\gamma,q} \times \underbrace{\mathbb{R}^q \times \cdots \times \mathbb{R}^q}_{p-1 \text{ times}}, \\ & \mathbf{W} \mapsto \mathbf{W} - \mathbf{B} \circ \mathbf{G}(\mathbf{W}), \end{aligned} \tag{15}$$

where $\mathbf{B}$ is an operator to be defined shortly and is “close” to $[d\mathbf{G}(\mathbf{W})]^{-1}$.

A heuristic evidence suggests that, for hyperbolic orbits, the above algorithm does indeed converge (for initial condition close enough to the ensuing solution). However, A formal proof of this claim can not be presented and is outside the scope of work for this paper. It should also be stated that this algorithm is expected not to converge in the case where the system undergoes certain bifurcations, e.g. fold bifurcations, heteroclinic bifurcations, etc. In the next section, we will explain how to estimate the differential $d\mathbf{G}(\mathbf{W})$ and how we choose $\mathbf{B}$. Computing such a differential amounts to computing the differential of the Step Map. To do so, we must estimate the Step Map of the variational equation associated with equation (13).

# 4 Differential of the Step Map

Recall that the Step Map accepts two inputs: $\alpha$ and the initial history data $\boldsymbol{\varphi} = \mathbf{J}^{-1}(\varphi)$. Our objective is to determine an operator $\mathbf{A}(\boldsymbol{\varphi})$ sufficiently close to $[d\mathbf{S}_{\alpha,q}(\varphi)]^{-1}$ in order to use the Newton-like method illustrated in Section 3.5. To do this, we calculate the differential of the Step Map when $\varepsilon = 0$.

To begin, the partial derivative of the Step Map with respect to $\alpha$ can be easily estimated thanks to the below observation. Recall that $\hat{x}_\varphi(t)$ denotes the $x$-component of the fixed-point of the Reduced Picard Operator, i.e.

$$\mathbf{S}_{\alpha,q}(\varphi)(t) = \hat{x}_\varphi(t+\alpha), \quad -\alpha \le t \le 0. \tag{16}$$

Assuming

$$\hat{x}_\varphi(u) = \sum_{j=0}^{q-1} p_j u^j, \quad \text{and} \quad \mathbf{S}_{\alpha,q}(\varphi)(u) = \sum_{j=0}^{q-1} S_j u^j,$$

we have that

$$\begin{pmatrix} S_0 \\ S_1 \\ \vdots \\ S_{q-1} \end{pmatrix} = \hat{\mathbf{T}} \begin{pmatrix} p_0 \\ p_1 \\ \vdots \\ p_{q-1} \end{pmatrix}, \tag{17}$$

where

$$\hat{\mathbf{T}} = \left(\hat{T}_{i,j}\right)_{1 \le i,j \le q-1} \text{ with } \hat{T}_{i,j} = \alpha^{j-i}\binom{j-1}{i-1} \text{ if } j \ge i, \ \hat{T}_{i,j} = 0 \text{ if } j < i. \tag{18}$$

From (16), we have that

$$\frac{\partial \mathbf{S}_{\alpha,q}(\varphi)}{\partial \alpha}(t) = \frac{d\hat{x}_{\varphi}}{dt}(t+\alpha). \tag{19}$$

With (17) and (18), we then have

$$\mathbf{J}^{-1}\left[\frac{\partial \mathbf{S}_{\alpha,q}(\varphi)}{\partial \alpha}\right] = \hat{\mathbf{T}} \begin{pmatrix} p_1 \\ 2p_2 \\ \vdots \\ (q-1)p_{q-1} \\ 0 \end{pmatrix}. \tag{20}$$

To estimate the other partial derivatives, we study the variational equation associated with (13) when $\varepsilon = 0$.

## 4.1 Variational Equation

We express the initial history function as

$$\mathbf{J}(\boldsymbol{\varphi}) = \varphi(t) = \sum_{j=0}^{q-1} \varphi_j t^j.$$

In what follows, we will use the notation

$$u_j(t) = \frac{\partial x}{\partial \varphi_j}(t), \quad v_j(t) = \frac{\partial y}{\partial \varphi_j}(t), \quad j = 0, \ldots, q-1.$$

For each integer $j = 0, \ldots, q-1$, we differentiate both sides of (13) with respect to $\varphi_j$ After doing so and setting $\varepsilon = 0$, we obtain the following system:

$$\left\{ \begin{array}{lcll} \dot{u}_j & = & \tau \mathcal{L}_{q-1}(v_j), & j = 0, \ldots, q-1 \\ \dot{v}_j & = & \tau \mathcal{L}_{q-1}\Big[\mu\Big(1 - \tilde{x}^2(t-1)\Big)v_j - u_j - 2\mu\tilde{x}(t-1)\tilde{u}_j(t-1)\hat{y}_{\varphi},\Big] & \end{array} \right. \tag{21}$$

where

$$\tilde{u}_j(w) = w^j, \ \text{ if } \ w < 0, \ \text{ and } \ \tilde{u}_j(w) = u_j(w), \ \text{ if } \ w \geq 0.$$

and where

$$\tilde{x}(u) = \varphi(u), \ \text{ if } \ u < 0, \ \tilde{x}(u) = \hat{x}_{\varphi}(u), \ \text{ if } \ u > 0.$$

Assume that, for each $j = 0, \ldots, q-1$, the above equation is solved. We then write the matrix

$$\boldsymbol{\Lambda}(\boldsymbol{\varphi}) = \Big(\Lambda_{j,k}(\boldsymbol{\varphi})\Big)_{j=1,\ldots,q,\ k=1,\ldots,q}$$

where for each $k = 0, \dots, q-1$,

$$u_k(t) = \sum_{j=1}^{q} \Lambda_{j,k+1}(\boldsymbol{\varphi}) t^{j-1}.$$

Thanks to (17) we have

$$\mathbf{J}^{-1}\left[\frac{\partial \mathbf{S}_{\alpha,q}(\varphi)}{\partial \varphi_k}\right] = \hat{\mathbf{T}} \begin{pmatrix} \Lambda_{1,k+1} \\ \Lambda_{2,k+1} \\ \vdots \\ \Lambda_{q-1,k+1} \\ \Lambda_{q,k+1} \end{pmatrix}. \tag{22}$$

and therefore, for each vector $\mathbf{w} \in \mathbb{R}^q$ we have

$$\mathbf{J}^{-1} \circ d\mathbf{S}_{\alpha,q}(\varphi) \circ \mathbf{J}(\mathbf{w}) \approx \mathbf{A}(\boldsymbol{\varphi}) \cdot \mathbf{w} \tag{23}$$

where

$$\mathbf{A}(\boldsymbol{\varphi}) \quad = \quad \hat{\mathbf{T}} \boldsymbol{\Lambda}(\boldsymbol{\varphi}). \tag{24}$$

Like before, we may solve Eq. (21) by seeing the corresponding solution as a fixed point of the following Picard operator:

$$\begin{aligned} \mathbf{V}_{q,j} &: \mathbb{P}_{q-1}[t] \times \mathbb{P}_{q-1}[t] \to \mathbb{P}_{q-1}[t] \times \mathbb{P}_{q-1}[t], \\ (u_j, v_j) &\mapsto \Big( \mathrm{V}_{q,j,1}(u_j, v_j), \mathrm{V}_{q,j,2}(u_j, v_j) \Big), \end{aligned} \tag{25}$$

where

$$\mathrm{V}_{q,j,1}(u_j, v_j) = u_j(0) + \tau \int_0^t \mathcal{L}_{q-1}(v_j(s)) ds$$

and

$$\mathrm{V}_{q,j,2}(u_j, v_j) = v_j(0) +$$

$$\tau \int_0^t \mathcal{L}_{q-1}\Big( \mu(1 - \tilde{x}^2(s-1)) v_j(s) - u_j(s) - 2\mu \tilde{x}(s-1) \tilde{u}_j(s-1) \hat{y}_\varphi(s) \Big) ds,$$

where

$$u_0(0) = 1, \quad u_j(0) = 0, \text{ if } \; j \geq 1.$$

Since a solution $(x(t), y(t))$ of (6) needs to satisfy $dx/dt = y(t)/\tau$, we then must have

$$v_0(0) = 0, \quad v_1(0) = 1/\tau, \quad \text{and} \quad v_j(0) = 0, \text{ if } \; j \geq 2.$$

## 4.2 Another Newton-like operator

To compute the desired fixed point, we again use the same Newton-like method, i.e. we construct the operator

$$\begin{aligned} \hat{\mathbf{N}}_{q,j}:\ \mathbb{P}_q[t]\times\mathbb{P}_q[t]\times &\to \mathbb{P}_q[t]\times\mathbb{P}_q[t], \\ (u_j,v_j)\mapsto (u_j,v_j) &- \hat{\mathbf{A}}\circ\Big(\mathbf{V}_{q,j}(u_j,v_j)-(u_j,v_j)\Big) \end{aligned} \tag{26}$$

where $\hat{\mathbf{A}}$ is an operator estimating the inverse of $d\mathbf{V}_{q,j}(u_j,v_j)-\mathbf{Id}$. Let $h,k\in\mathbb{P}_q[t]$. When setting $\varepsilon=0$, a straightforward computation shows that, for any $j=0,\dots,q-1$, we have

$$d\mathbf{V}_{q,j}(u_j,v_j)(h,k)=\Big(L_x(h,k),L_y(h,k)\Big),$$

where

$$\begin{aligned} L_x(h,k) &= \tau\int_0^t \mathcal{L}_{q-1}(k(s))ds, \\ L_y(h,k) &= \tau\int_0^t \mathcal{L}_{q-1}\Big(\mu(1-\tilde{x}^2(s-1))k(s)-h(s) \\ &\qquad -2\mu\tilde{x}(s-1)\tilde{h}_j(s-1)\hat{y}_\varphi(s)\Big)ds \end{aligned}$$

where

$$\tilde{h}(w)=0,\quad\text{if}\quad w<0,\quad\text{and}\quad \tilde{h}(w)=h(w),\quad\text{if}\quad w\geq 0.$$

Observe that this differential does not depend upon the choice of $j$. We will need to express this differential in the standard base, i.e., writing

$$h(t)=\sum_{s=0}^{q-1}h_st^s,\quad h(t)=\sum_{s=0}^{q-1}h_st^s,$$

or equivalently

$$h=\mathbf{J}\begin{pmatrix}h_1\\ \vdots\\ h_{q-1}\end{pmatrix},\quad k=\mathbf{J}\begin{pmatrix}k_1\\ \vdots\\ k_{q-1}\end{pmatrix},$$

we have

$$\begin{pmatrix}\mathbf{J}^{-1}(L_x(h,k))\\ \\ \mathbf{J}^{-1}(L_y(h,k))\end{pmatrix}=\hat{\mathbf{B}}\begin{pmatrix}h_1\\ \vdots\\ h_{q-1}\\ k_1\\ \vdots\\ k_{q-1}\end{pmatrix}$$

where

$$\hat{\mathbf{B}} = \begin{pmatrix} \mathbf{O}_q & \hat{\mathbf{B}}_2 \\ \hat{\mathbf{B}}_3 & \hat{\mathbf{B}}_4 \end{pmatrix}$$

and where $\mathbf{0}_q$ is the $q \times q$ matrix with 0 entries. Furthermore, for each $\ell = 2, 3, 4$, if we denote by $\hat{\mathbf{B}}_{\ell,r}$ the $r^{\text{th}}$ column of $\hat{\mathbf{B}}_\ell$ we have

$$\hat{\mathbf{B}}_{2,r} = \mathbf{J}^{-1}\Big(\tau \int_0^t \mathcal{L}_{q-1}(s^{r-1})ds\Big),$$

$$\hat{\mathbf{B}}_{3,r} = \mathbf{J}^{-1}\Bigg(\tau \int_0^t \mathcal{L}_{q-1}\Big(-2\mu\tilde{x}(s-1)\hat{y}_\varphi(s)w_r(s-1)) - s^{r-1}\Big)ds\Bigg),$$

where for $r > 1$,

$$w_r(u) = u^{r-1}, \ \text{ if } u \geq 0, \quad w_r(u) = 0 \ \text{ if } u < 0.$$

If we apply the above definition for $r = 1$, the corresponding function $w_1$ is discontinuous and therefore one encounters a Gibbs-like effect. To avoid this issue, we write

$$\hat{\mathbf{B}}_{3,1} = -\mathbf{J}^{-1}(\tau t) = -\begin{pmatrix} 0 \\ \tau \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \text{if } \ t \leq 1,$$

and

$$\hat{\mathbf{B}}_{3,1} = \mathbf{J}^{-1}\Big(\tau \int_1^t \tilde{\mathcal{L}}_{q-1}(-2\mu\tilde{x}(s-1)\hat{y}_\varphi(s))ds\Big) - \mathbf{J}^{-1}(\tau t), \ \text{ if } \ t \geq 1,$$

where $\tilde{\mathcal{L}}_{q-1}$ is defined in the former section. Finally we have

$$\hat{\mathbf{B}}_{4,r} = \mathbf{J}^{-1}\Bigg(\tau \int_0^t \mathcal{L}_{q-1}\Big(\mu(1 - \tilde{x}^2(s-1))s^{r-1}\Big)ds\Bigg).$$

In what follows, $\mathbf{Id}_{2q}$ designates the $2q \times 2q$ identity matrix. We define the operator $\hat{\mathbf{A}}$ as follows:

$$\hat{\mathbf{A}} = \mathbf{J} \circ \Big(\hat{\mathbf{B}} - \mathbf{Id}_{2q}\Big)^{-1} \circ \mathbf{J}^{-1}.$$

### 4.3 Construction of B

The differential $d\mathbf{G}(a_1^*, \mathbf{a}_2, \ldots, \mathbf{a}_p)$ writes

$$d\mathbf{G}(a_1^*, \mathbf{a}_2, \ldots, \mathbf{a}_p)(w) = \mathbf{M} \cdot w$$

where

$$\mathbf{M} = \begin{pmatrix} \mathbf{A}_1^* & -\mathrm{Id}_q & \mathbf{0}_q & \mathbf{0}_q & \cdots & \cdots & \mathbf{0}_q \\ \mathbf{0}_{q,2}^* & \mathbf{A}(\mathbf{a}_2) & -\mathrm{Id}_q & \mathbf{0}_q & \cdots & \cdots & \mathbf{0}_q \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{q,j}^* & \cdots & \mathbf{0}_q & \mathbf{A}(\mathbf{a}_j) & -\mathrm{Id}_q & \cdots & \mathbf{0}_q \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{q,p-1}^* & \cdots & \cdots & \cdots & \mathbf{0}_q & \mathbf{A}(\mathbf{a}_{p-1}) & -\mathrm{Id}_q \\ -\mathrm{Id}_q^* & \mathbf{0}_q & \cdots & \cdots & \cdots & \mathbf{O}_q & \mathbf{A}(\mathbf{a}_p) \end{pmatrix}$$

where $\mathbf{O}_q$ and $\mathrm{Id}_q$ are respectively the zero matrix and the identity matrix, both of dimension $q$, and the matrices

$$\mathbf{A}(\mathbf{a}_\ell), \quad \ell = 2, \ldots, p-1$$

are defined thanks to (24). Furthermore, every column of the matrix $\mathbf{A}_1^*$ coincides with those of $\mathbf{A}(\mathbf{a}_1)$, except for the first columns of $\mathbf{A}_1^*$. This latter is given by the vector

$$\mathbf{J}^{-1}\Big(\frac{\partial \mathbf{S}_{\alpha,q}(\mathbf{J}^*(\mathbf{a}_1^*))}{\partial \alpha}\Big) = \begin{pmatrix} w_{1,1} \\ 2w_{1,2} \\ \vdots \\ (q-1)w_{1,q-1} \\ 0 \end{pmatrix},$$

$$\text{where } \mathbf{J}\left(\begin{pmatrix} w_{1,0} \\ w_{1,1} \\ \vdots \\ w_{1,q-1} \end{pmatrix}\right) = \mathbf{S}_{\alpha,q}\Big(\mathbf{J}^*(\mathbf{a}_1^*)\Big).$$

Similarly, for each integer $2 \leq j \leq p-1$, every column of the matrix $\mathbf{O}_{q,j}^*$ coincides with those of $\mathbf{O}_q$, except for the first columns of $\mathbf{O}_{q,j}^*$. This latter is given by the vector

$$\mathbf{J}^{-1}\Big(\frac{\partial \mathbf{S}_{\alpha,q}(\mathbf{J}(\mathrm{a}_j))}{\partial \alpha}\Big) = \begin{pmatrix} w_{j,1} \\ 2w_{j,2} \\ \vdots \\ (q-1)w_{j,q-1} \\ 0 \end{pmatrix},$$

$$\text{where}\quad \mathbf{J}\left(\begin{pmatrix} w_{j,0} \\ w_{j,1} \\ \vdots \\ w_{j,q-1} \end{pmatrix}\right) = \mathbf{S}_{\alpha,q}\Big(\mathbf{J}(\mathbf{a}_j)\Big),$$

Finally, $-\mathrm{I}d_q^*$ coincides with $-\mathrm{I}d_q$, except that the first column is given by the vector

$$\mathbf{J}^{-1}\left(\frac{\partial \mathbf{S}_{\alpha,q}(\mathbf{J}(\mathbf{a}_p))}{\partial \alpha}\right) = \begin{pmatrix} w_{p,1} \\ 2w_{p,2} \\ \vdots \\ (q-1)w_{p,q-1} \\ 0 \end{pmatrix},$$

$$\text{where}\quad \mathbf{J}\left(\begin{pmatrix} w_{p,0} \\ w_{p,1} \\ \vdots \\ w_{p,q-1} \end{pmatrix}\right) = \mathbf{S}_{\alpha,q}\Big(\mathbf{J}(\mathbf{a}_{m-1})\Big).$$

# 5 Implementation

To conclude, we describe how the orbits obtained from the methods described in the previous section can be computed within a given tolerance. Furthermore, we illustrate in Section 5.2 one method to solve the state-independent case, i.e. when $\epsilon = 0$. Doing this allows us to generate initial conditions to use in the Newton-like method described above. Using these as seed values, one may generate periodic orbits for systems with $\epsilon \neq 0$, but small in magnitude. Then, one may use those orbits to generate periodic orbits for slightly larger values of $\epsilon$, and so on. This technique of course does not yield all periodic orbits for a given set of parameters.

## 5.1 *A posteriori* check

We choose $\delta > 0$ a small real number to be the tolerance in our computation. Let $1 < \alpha < \alpha_0$, (in our computation, we choose $\alpha_0 = 1.4$). Thanks to Lemma 1, there exists $q_0 = q_0(b) > 1$ such that for all $q \geq q_0$ and for all $z \in C_b^1([0, \alpha_0])$

$$\sup_{0 \leq t \leq \alpha_0} |\mathcal{L}_{q-1}(z) - z| \leq \delta/3.$$

Let $q > q_0$ and $\varphi \in \mathbb{P}_q$ that satisfies

$$\sup_{-\alpha \leq t \leq 0} |\varphi(t)| \leq M.$$

We compute $S_{\alpha,q}(\varphi)$ as follows. Assuming

$$\varphi(t) = \sum_{j=0}^{q} \varphi_j t^j,$$

we define our ‘initial guess’

$$\mathbf{f}_0(t) = (\varphi_0 + \varphi_1 t, \varphi/\tau),$$

and construct the sequence

$$\mathbf{f}_{k+1} = \hat{\Psi}_{\varphi,q}(\mathbf{f}_k), \quad k = 0, \dots.$$

We choose $\mathbf{n} > 1$ such that for all $p > 0$

$$\|\mathbf{f}_{\mathbf{n}+p}(t) - \mathbf{f}_{\mathbf{n}}(t)\| \le \delta/2, \quad \forall\ 0 \le t \le \alpha \ \text{ and } \ \forall\ p > 0. \tag{27}$$

This implies that

$$\|\hat{\mathbf{z}}(t) - \mathbf{f}_{\mathbf{n}}(t)\| \le \delta/2, \quad \forall\ 0 \le t \le \alpha, \tag{28}$$

where $\hat{\mathbf{z}}$ is defined in Theorem 2. Thanks to Theorem 2, we choose $q \ge \max\{q_1, q_0\}$ such that

$$\|\hat{\mathbf{z}}(t) - \mathbf{z}(t)\| \le \delta/2, \quad \forall\ 0 \le t \le \alpha, \tag{29}$$

and with (28) it follows that

$$\|\mathbf{z}(t) - \mathbf{f}_{\mathbf{n}}(t)\| \le \delta, \quad \forall\ 0 \le t \le \alpha. \tag{30}$$

We also verify that

$$\left\| \frac{d\mathbf{f}_{\mathbf{n}}}{dt} - \tau \mathbf{F}(\mathbf{f}_{\mathbf{n}}(t), \mathbf{f}_{\mathbf{n}}(t - 1 + \varepsilon \mathbf{f}_{\mathbf{n},x}(t)), \right\| \le \delta, \quad \forall\ 0 \le t \le \alpha \tag{31}$$

(where $\mathbf{f}_{\mathbf{n}} = (\mathrm{f}_{\mathbf{n},x}, \mathrm{f}_{\mathbf{n},y})$ ).

## 5.2 State-Independent Delay Equation

To conclude, we illustrate one method to find periodic solutions of the state-independent case of the DDE (6), i.e. when $\varepsilon = 0$. Our motivation for solving the state-independent case is to have data to be used as an adequate initial condition for the Newton-like procedure described in Section 3. Specifically, we study the system of the form

$$\begin{cases} \dot{\mathbf{x}}(t) = \tau \mathbf{F}\left(\mathbf{x}(t), \mathbf{x}(t-1)\right) \\ \mathbf{x}(t) = \phi(t), \qquad t \in [-1, 0] \end{cases} \tag{32}$$

where $x(t) \in \mathbb{R}^n$, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$, and $\phi(t) \in \mathbb{R}^n$ is a known function. Furthermore, we assume that $\tau > 0$ is a known parameter, obtained from the (unperturbed) delay of the original state-dependent equation.

Since our search is for periodic solutions of (32), we can reasonably propose a Fourier series ansatz solution of the form

$$\tilde{\mathbf{x}}(t) = \sum_{k \in \mathbb{Z}} \mathbf{a}_k e^{\omega_k t}, \tag{33}$$

where $\mathbf{a}_k \in \mathbb{R}^n$ for all $k \in \mathbb{Z}$ and $\omega_k := \dfrac{2\pi i k}{T}$, $T$ being the (unknown) period of $\tilde{\mathbf{x}}$. A rigorous treatment of the method to follow, as well as an outline for computer-assisted proofs of the existence and uniqueness of periodic solutions in similar systems, can be found in [35, 16]. In the case of the delayed Van der Pol system,

$$\begin{cases} \dot{x}(t) = \tau y(t), \\ \dot{y}(t) = \tau\mu\left(1 - x^2(t-1)\right) y(t) - \tau x(t) \end{cases}, \tag{34}$$

plugging in the ansatz (33) with $\mathbf{a}_k := \begin{pmatrix} a_k \\ b_k \end{pmatrix}$ yields the relations

$$\begin{cases} \displaystyle\sum_{k\in Z} \omega_k a_k e^{\omega_k t} = \tau \sum_{k\in\mathbb{Z}} b_k e^{\omega_k t} \\ \displaystyle\sum_{k\in\mathbb{Z}} \omega_k b_k e^{\omega_k t} = \tau\mu \left( \sum_{k\in\mathbb{Z}} b_k e^{\omega_k t} - \sum_{k\in\mathbb{Z}} c_k e^{\omega_k t} \right) - \tau \sum_{k\in\mathbb{Z}} a_k e^{\omega_k t}. \end{cases} \tag{35}$$

where $c_k$ is to be defined below. Recall the discrete convolution $*$, defined by

$$(a * a)_k = \sum_{j\in\mathbb{Z}} a_{k-j} a_j.$$

Hence, $c_k$ in equation (35) is defined as

$$\begin{aligned} c_k &:= \left( \left\{ a_j e^{-\omega_j} \right\}_{j\in\mathbb{Z}} * \left\{ a_j e^{-\omega_j} \right\}_{j\in\mathbb{Z}} * \left\{ b_j \right\}_{j\in\mathbb{Z}} \right)_k \\ &= \sum_{j\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} a_{k-j} a_{j-\ell} b_\ell e^{-(\omega_{k-j} + \omega_{j-\ell})}. \end{aligned}$$

Finally, after bringing the sums to one side and setting them equal to zero, we conclude that the ansatz will be a solution to the delayed Van der Pol system (34) if and only if

$$\begin{cases} \omega_k a_k - \tau b_k & = 0, \\ \tau a_k + (\omega_k - \tau\mu) b_k + \tau\mu c_k & = 0, \end{cases}$$

for all $k \in \mathbb{Z}$.

In order to find such a sequence that satisfies the relations above, we must formulate the proper functional equations to solve, as well as define a proper space in which to solve them. As System (34) is analytic, its periodic solutions must also be analytic [24]. Hence, we consider Fourier coefficients in the space

$$\ell^1_\nu = \left\{ \{a_k\}_{k\in\mathbb{Z}} \in \mathbb{C}^{\mathbb{Z}} : a_k = \overline{a_{-k}} \quad \text{and} \quad \sum_{k\in\mathbb{Z}} |a_k| \nu^{|k|} < \infty \right\},$$

where $\nu > 1$ is a fixed real number. The conditions of the space $\ell^1_\nu$ above for $\nu > 1$ ensure that the Fourier series expansion (33) will be well-defined, analytic, and real-valued for all $t \in \mathbb{R}$. Furthermore, the space $\ell^1_\nu$ forms a Banach algebra under the discrete convolution product $*$ defined above.

For each $k \in \mathbb{Z}$, define the map $F_k : (\ell^1_\nu)^2 \times \mathbb{R} \to \mathbb{R}^2$ by

$$F_k(\mathbf{a}, T) = \begin{pmatrix} \omega_k a_k - \tau b_k \\ \tau a_k + (\omega_k - \tau\mu) b_k + \tau\mu c_k \end{pmatrix}.$$

Concatenating these maps allows us to define the operator $\mathbf{F} : (\ell^1_\nu)^2 \times \mathbb{R} \to (\ell^1_{\nu_0})^2$:

$$\mathbf{F}(\mathbf{a}, T) = \{F_k(\mathbf{a}, T)\}_{k \in \mathbb{Z}},$$

where the image space depends on $\nu_0 < \nu$, as $\{\omega_k a_k\}$ is not guaranteed to converge in $\ell^1_\nu$. See [16] for more details and a proof that $\nu_0$ exists.

We note that finding zeroes of $\mathbf{F}$ above allows us to discover periodic solutions of (34). However, we must still consider the phase condition $\mathbf{x}(t) = \phi(t)$ for $t \in [-1, 0]$ given in (32). To achieve this, we define the map $G_0(\mathbf{a}, T) : (\ell^1_\nu)^2 \times \mathbb{R} \to \mathbb{R}$ by

$$G_0(\mathbf{a}, T) = \left( \sum_{k \in \mathbb{Z}} a_k \right) - \phi_1(0),$$

where $\phi_1(t)$ is the first component of $\phi(t)$. Finally, we concatenate our maps and study the operator $\mathbf{G} : \left(\ell^1_\nu\right)^2 \times \mathbb{R} \to \left(\ell^1_{\nu_0}\right)^2 \times \mathbb{R}$, defined by

$$\mathbf{G}(\mathbf{a}, T) = \begin{pmatrix} \mathbf{F}(\mathbf{a}, T) \\ G_0(\mathbf{a}, T) \end{pmatrix}$$

It can be seen that zeroes of the above map $\mathbf{G}$ correspond to solutions of (34) via the isomorphism $\mathbf{a} \mapsto \sum_{k \in \mathbb{Z}} \mathbf{a}_k e^{\omega_k t}$, and it would be natural to define the associated Newton-like operator $\mathbf{H} : \left(\ell^1_\nu\right)^2 \times \mathbb{R} \to \left(\ell^1_{\nu_0}\right)^2 \times \mathbb{R}$ defined by

$$\mathbf{H}(\mathbf{z}) \coloneqq \mathbf{z} - \left( D\mathbf{G}^{-1} \circ \mathbf{G} \right)(\mathbf{z}).$$

We first note here, however, that like in the sections above, it is infeasible to calculate the true action of $D\mathbf{G}^{-1}$, and thus this map will be replaced with a nearby approximation in the Newton-like approach, detailed below. Furthermore, as the image of $\mathbf{H}$ in the way it is defined above is not the same as its domain, a contraction argument cannot be immediately applied to find fixed points of $\mathbf{H}$.

We then proceed by numerically approximating fixed-points of $\mathbf{H}$ by truncating all of the maps above and numerically approximating $D\mathbf{G}^{-1}$ in finite-dimensional space. A rigorous analysis of this exact method is common in computer-assisted proofs, and the reader is again referred to sources such as [35, 16]. As we are only interested in using the generated solutions to (34) as initial data for the Newton-like operator in Section 3, we forego the analysis complete and continue with a heuristic approach.

In order to obtain a periodic solution of (34), one may choose values of $\tau$ and $\mu$ such that there exists a numerically stable attracting periodic orbit. (Examples of orbits with such values of $\tau$ and $\mu$ are illustrated in Figures 1 - 3 below). Once a periodic orbit is numerically obtained and $T$ is estimated with $\hat{T}$, one may deduce the initial sequence-valued data $\hat{\mathbf{a}}$ for the truncated Newton-like operator $\hat{\mathbf{H}}$ by numerically integrating the orbit on an interval of length $\hat{T}$. Iterating the process yields an approximate periodic solution of (34). The resulting data may then be used as initial conditions for the multiple-shooting, Newton-like method described earlier and was often used to generate the illustrations below.

One may then use these outputs for the $\epsilon = 0$ Newton-like method as suitable initial conditions for a new Newton-like method for parameters $\tau + \delta_1$ and $\mu + \delta_2$, where $\delta_1 > 0$ and $\delta_2 > 0$ are sufficiently small. Iterating this process yields sufficient data of periodic orbits of the state-independent system for many different choices of the parameters. These data may then be fed as initial conditions to the Newton-like methods described in Section 3. It should be noted that it is not feasible for this method to robustly describe all periodic orbits for a given set of parameters whether in the state-dependent or fixed delay case.

By nature of the Newton-like method employed, the convergence of this algorithm can only be achieved for hyperbolic periodic orbits. Similarly, we do not observe (nor do we expect) this algorithm to converge nearby saddle-node bifurcations, as the derivative $D\mathbf{G}$ is degenerate at these values.

### 5.3 Figures & Conclusion

We present below some illustrations of periodic orbits calculated for various choices of $\tau, \mu$, and $\epsilon$, as well as examples of typical orbits in this system with the given parameters. To generate these select periodic orbits, we used a fixed tolerance of $\delta = 10^{-8}$, which was achieved with the number of Lagrange-Chebyshev nodes set to $q = 10$. A tighter tolerance would require a higher number of nodes, which would ultimately increase computational runtime of this algorithm. However, we are not necessarily interested in high-precision periodic solutions, and so the choices of $\nu$ and $q$ above suffice for the scope of this paper.

The methods described above were highly successful in generating periodic orbits of a state-dependent Van der Pol system. The authors feel that a generalization of this method for higher-dimensional problems can be achieved using similar methods. Of more interest, however, would be the natural extension of this work, namely framing the results of Section 4 in the context of computer-assisted proof. In particular, following closely the techniques found in [16, 35] and [13] (this latter already framed in the context of state-dependent DDEs), one could derive rigorous, validated numerics of the Newton-like method described above and formulate proofs of the existence and persistence of such orbits. Such an analysis is beyond the scope of this present work, but would make a thoroughly interesting future project.

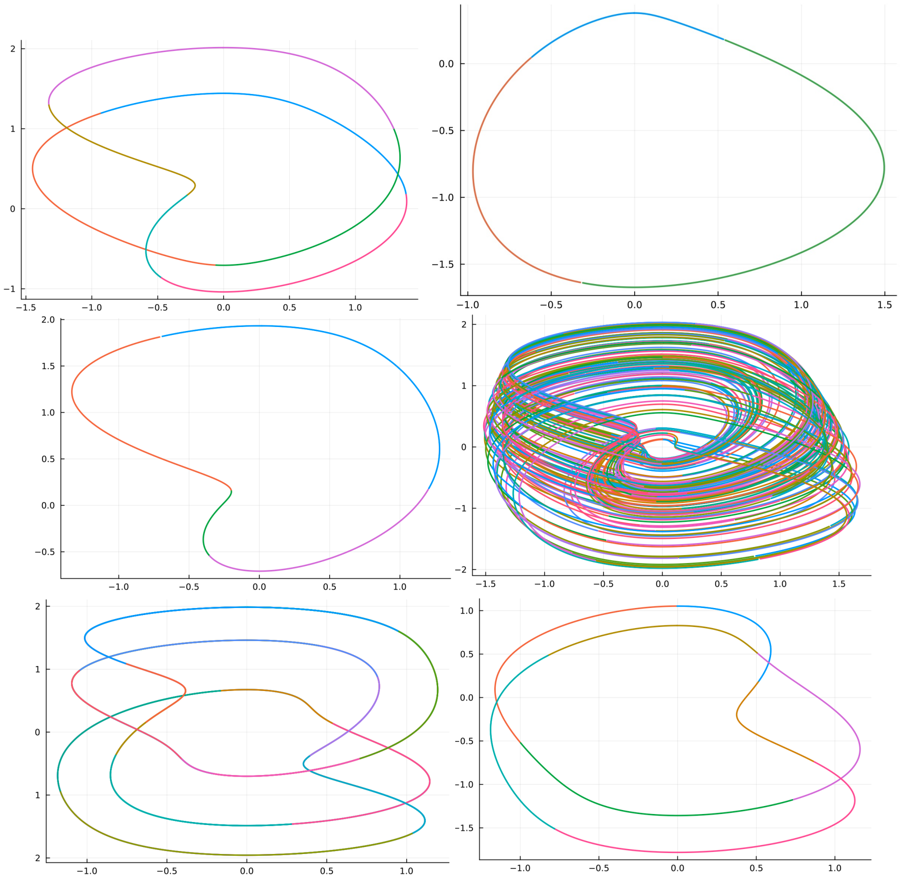

Figure 1: Top and Middle Left: Coexisting (Lyapunov unstable) periodic orbits (in the $(x, y)$-plane) when $\tau = 1.8,\ \mu = 0.95,\ \varepsilon = 0.03$.
Middle Right: A typical orbit.
Bottom Left and Right: Coexisting (Lyapunov stable and unstable, respectively) periodic orbits when $\tau = 1.5,\ \mu = 0.95,\ \varepsilon = 0.02$.

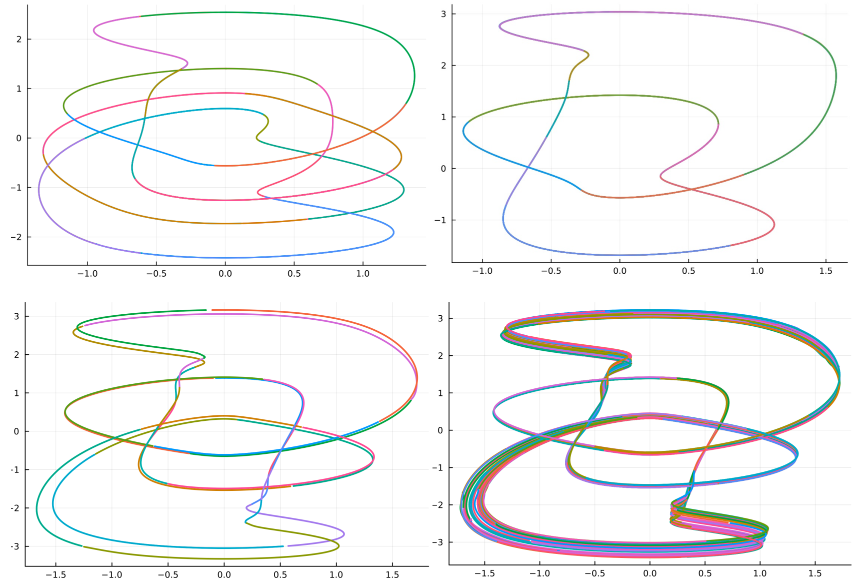

Figure 2: Top: Two Lyapunov stable periodic orbits when (left) $\tau = 1.63,\ \mu = 1.1,\ \varepsilon = 0.03$ and (right) $\tau = 1.58,\ \mu = 1.2,\ \varepsilon = 0.012$.

Bottom left: A Lyapunov unstable periodic orbit when $\tau = 1.8,\ \mu = 1.25,\ \varepsilon = -0.0152$.

Bottom Right: A typical orbit for the same value of the parameter.

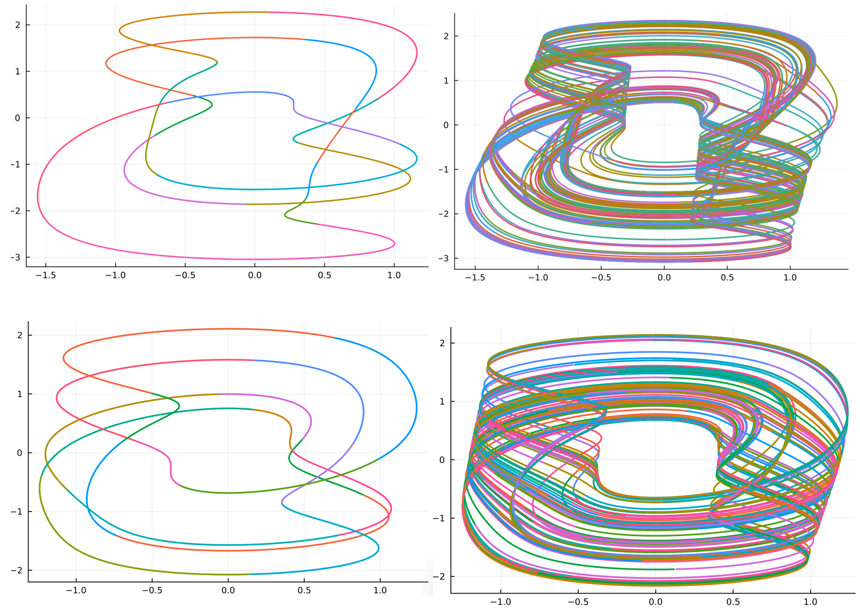

Figure 3: Top left: A Lyapunov stable periodic orbit when $\tau = 1.59, \mu = 1.21,\ \varepsilon = 0.015$.
Top Right: A more complicated orbit for the same parameter values.
Bottom left: A Lyapunov stable periodic orbit when $\tau = 1.51,\ \mu = 1.21,\ \varepsilon = 0.015$.
Bottom Right: a more complicated orbit for the same parameter values.

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# 6 Appendix

In this appendix we give the details for the proofs of Theorem 1 and 2.

## Proof of Theorem 1

PROOF: In what follows, we have, for $i = 1, 2$,

$$\tilde{f}_{i,x}(u) = f_{i,x}, \ \text{ if } u \geq 0, \quad \text{ and } \quad \tilde{f}_{i,x}(u) = \varphi(u) \ \text{ if } u < 0.$$

We first observe that, with the choice of $b$ made in (9), we can easily verify that for all $\beta > 0$, and for all $0 \leq t \leq \beta$

$$\left\| \frac{d}{dt}\Big( \mathbf{\Psi}_{\varphi,\beta}(\mathbf{z})(t) \Big) \right\| \leq \tau \|\mathbf{F}(\mathbf{z}(t), \tilde{x}(t - 1 + \varepsilon x(t)\| \leq b.$$

Furthermore, we have

$$\begin{aligned} ||\mathbf{\Psi}_{\varphi,\beta}(\mathbf{z})(t)|| &\leq \left\| \begin{pmatrix} \varphi(0) \\ \\ \dfrac{\dot{\varphi}(0)}{\tau} \end{pmatrix} \right\| + \tau\beta \sup_{0 \leq t \leq \beta} \|\mathbf{F}(\mathbf{z}(t), \tilde{x}(t - 1 + \varepsilon x(t)\| \\ &\leq \mathrm{M}/2 + \beta b. \end{aligned}$$

This implies that if $\beta \leq \beta_1 := \mathrm{M}/2b$, then $\mathbf{\Psi}_{\varphi,\beta}$ leaves $\mathbf{S}_{\varphi,\beta}$ invariant. The remaining statement of Theorem 1 is a consequence of the following lemma.

**Lemma 2** *There exists* $0 < \beta_0 \leq \beta_1$ *such that, for all* $\beta \leq \beta_0$, $a \geq 0$, $t \geq 0$, *and* $\mathbf{f}_1$, $\mathbf{f}_2 \in \mathbf{C}_b^1([0, a + \beta])$,

$$\begin{aligned} &\int_a^{a+t} \left\| \tau \Big[ \mathbf{F}(\mathbf{f}_1(s), f_{1,x}(s - 1 + \varepsilon \tilde{f}_{1,x}(s))) - \mathbf{F}(\mathbf{f}_2(s), f_{2,x}(s - 1 + \varepsilon \tilde{f}_{2,x}(s))) \Big] \right\| ds \\ &\leq \frac{1}{2} \mathbf{d}_{a+\beta}(\mathbf{f}_1, \ \mathbf{f}_2). \end{aligned} \tag{36}$$

PROOF: Recall that

$$\mathbf{F}(\mathbf{z}, u) = \begin{pmatrix} y \\ \mu(1 - u^2)y - x \end{pmatrix}, \quad \text{where } \mathbf{z} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

We write

$$\begin{aligned} &\mathbf{F}(\mathbf{f}_1(s), f_{1,x}(s - 1 + \varepsilon \tilde{f}_{1,x}(s))) - \mathbf{F}(\mathbf{f}_2(s), f_{2,x}(s - 1 + \varepsilon \tilde{f}_{2,x}(s))) \\ &= \mathbf{J}(s) + \mathbf{W}(s), \end{aligned} \tag{37}$$

where

$$\mathbf{J}(s)=\mathbf{F}(\mathbf{f}_1(s),f_{1,x}(s-1+\varepsilon\tilde{f}_{1,x}(s))-\mathbf{F}(\mathbf{f}_1(s),f_{2,x}(s-1+\varepsilon\tilde{f}_{2,x}(s)),$$

and

$$\mathbf{W}(s)=\mathbf{F}(\mathbf{f}_1(s),f_{2,x}(s-1+\varepsilon\tilde{f}_{2,x}(s))-\mathbf{F}(\mathbf{f}_2(s),f_{2,x}(s-1+\varepsilon\tilde{f}_{2,x}(s)).$$

A straightforward computation reveals

$$\mathbf{W}(s)=\left(\begin{array}{l} f_{1,y}(s)-f_{2,y}(s) \\ \\ \mu\mathbf{U}(s)[f_{1,y}(s)-f_{2,y}(s)]+f_{2,x}(s)-f_{1,x}(s)\end{array}\right) \tag{38}$$

where

$$\mathbf{U}(s)=[1-f_{2,x}^2(s-1+\varepsilon\tilde{f}_{2,x}(s)))],$$

and

$$\mathbf{J}(s)=\left(\begin{array}{l} 0 \\ \\ \mu f_{1,y}(s)[f_{2,x}^2(s-1+\varepsilon\tilde{f}_{2,x}(s))-f_{1,x}^2(s-1+\varepsilon\tilde{f}_{1,x}(s))]\end{array}\right) \tag{39}$$

Since $\|\mathbf{f}_i\|\le \mathrm{M}\quad(i=1,2)$, we have

$$|\mathbf{U}(s)|\le 1+\mathrm{M}^2,\quad 0\le s\le a+\beta,$$

and thus

$$\|\mathbf{W}(s)\|\le(\mu(1+\mathrm{M}^2)+1)\mathbf{d}_{a+\beta}(\mathbf{f}_1,\mathbf{f}_2),\quad 0\le s\le a+\beta. \tag{40}$$

Observe also that

$$\begin{aligned} &|f_{2,x}^2(s-1+\varepsilon\tilde{f}_{2,x}(s))-f_{1,x}^2(s-1+\varepsilon\tilde{f}_{1,x}(s))| \\ \le\ &|f_{2,x}^2(s-1+\varepsilon\tilde{f}_{2,x}(s))-f_{2,x}^2(s-1+\varepsilon\tilde{f}_{1,x}(s))]| \\ +&|f_{2,x}^2(s-1+\varepsilon\tilde{f}_{1,x}(s))-f_{1,x}^2(s-1+\varepsilon\tilde{f}_{1,x}(s))]|.\end{aligned}$$

Using the Mean Value Theorem, we see that

$$|f_{2,x}^2(s-1+\varepsilon\tilde{f}_{2,x}(s))-f_{2,x}^2(s-1+\varepsilon\tilde{f}_{1,x}(s))]|\le 2\mathrm{M}b\varepsilon d_{a+\beta}(\mathbf{f}_1,\mathbf{f}_2)$$

and, by factoring,

$$|f_{2,x}^2(s-1+\varepsilon\tilde{f}_{1,x}(s))-f_{1,x}^2(s-1+\varepsilon\tilde{f}_{1,x}(s))]|\le 2\mathrm{M}d_{a+\beta}(\mathbf{f}_1,\mathbf{f}_2).$$

Recalling (39), we have

$$\|\mathbf{J}(s)\|\le 2\mu\mathrm{M}^2(1+b\varepsilon)d_{a+\beta}(\mathbf{f}_1,\mathbf{f}_2).$$

Combining the results of (37) and (40), we have

$$\begin{aligned}&\left\|\mathbf{F}(\mathbf{f}_1(s), f_{1,x}(s-1+\varepsilon\tilde{f}_{1,x}(s))) - \mathbf{F}(\mathbf{f}_2(s), f_{2,x}(s-1+\varepsilon\tilde{f}_{2,x}(s)))\right\| \\ &\le [2\mu\mathrm{M}^2(1+b\varepsilon) + (\mu(1+\mathrm{M}^2)+1)]\mathbf{d}_{a+\beta}(\mathbf{f}_1,\mathbf{f}_2), \quad 0 \le s \le a+\beta.\end{aligned} \tag{41}$$

We now choose $\beta_0$ such that

$$\beta_0\tau[2\mu\mathrm{M}^2(1+b\varepsilon) + (\mu(1+M^2)+1)] \le 1/2$$

and (36) follows.

## Proof of Theorem 2

Before presenting the proof of Theorem 2, we must note the following lemma.

**Lemma 3** *The sequence* $(a_n)$ *defined by*

$$a_1 = 1, \quad a_2 = 2, \qquad a_{n+1} = 2 + \sum_{j=2}^{n} a_j \text{ when } n \ge 2$$

*satisfies*

$$a_n = 2^{n-1}, \quad n \ge 1.$$

The proof of this follows immediately from induction on $n$.

Proof of Theorem 2. Let $\varphi \in C_b^1([-\alpha, 0])$ and $\mathbf{z} \in \mathbf{S}_{\varphi,\alpha}$. Let $L \ge 1$ be such that

$$L\beta_0 \le \alpha_0 < (L+1)\beta_0,$$

where $\beta_0$ is given in Lemma 2. Recall the function

$$\mathbf{H}_{\varphi,\mathbf{z}} : [0, \alpha_0] \to \mathbb{R}^2, \; t \mapsto \begin{pmatrix} \mathrm{H}_1(t) \\ \mathrm{H}_2(t) \end{pmatrix} = \tau\mathbf{F}(\mathbf{z}(t), x(t-1+\varepsilon\tilde{x}(t))).$$

One can verify that, for all $0 < t < \alpha_0$,

$$|\mathrm{H}_1'(t)| \le \tau b, \quad |\mathrm{H}_2'(t)| \le \ell$$

where

$$\ell = \tau\mu\left(2Mb(1+\epsilon b)\right)M + \mu(1+M^2)b + b.$$

Due to Lemma 1, given $\delta > 0$, there exists $q_1 = q_1(\delta) > 1$ such that

$$||\mathcal{L}_q(\mathbf{H}) - \mathbf{H}||_{\alpha_0} \le \tilde{\delta}, \tag{42}$$

for all $q > q_1$, where $\tilde{\delta} = \delta/2^{L+1}$. Let $\mathbf{z}_\varphi$ and $\hat{\mathbf{z}}_\varphi$ be as in the statement of Theorem 2 and $v \coloneqq \begin{pmatrix} \varphi(0) \\ \dot{\varphi}(0)/\tau \end{pmatrix}$. We have

$$\begin{aligned} \mathbf{z}_\varphi - v &= \int_0^t \tau \mathbf{F}(\mathbf{z}_\varphi(s), x(s-1+\varepsilon \tilde{x}(s)))ds \qquad (43) \\ \hat{\mathbf{z}}_\varphi - v &= \int_0^t \mathcal{L}_q\Big(\tau \mathbf{F}(\hat{\mathbf{z}}_\varphi(s), \hat{x}(s-1+\varepsilon \tilde{\hat{x}}(s)))\Big)ds, \qquad (44) \end{aligned}$$

recalling that

$$\begin{aligned} &\tilde{x}(u) = x(u) \ \text{ if } u \geq 0, \quad \tilde{x}(u) = \varphi(u) \ \text{ if } u < 0, \\ &\tilde{\hat{x}}(u) = \hat{x}(u) \ \text{ if } u \geq 0, \quad \tilde{\hat{x}}(u) = \varphi(u) \ \text{ if } u < 0. \end{aligned}$$

Subtracting (44) from (43), it follows that

$$\hat{\mathbf{z}}_\varphi - \mathbf{z}_\varphi = \int_0^t \Big(\mathbf{G}_1(s) + \mathbf{G}_2(s)\Big)ds \qquad (45)$$

where

$$\mathbf{G}_1(s) = \mathcal{L}_q\Big(\tau \mathbf{F}(\hat{\mathbf{z}}_\varphi(s), \hat{x}(s-1+\varepsilon \tilde{\hat{x}}(s)))\Big) - \tau \mathbf{F}(\hat{\mathbf{z}}_\varphi(s), \hat{x}(s-1+\varepsilon \tilde{\hat{x}}(s))$$

and

$$\mathbf{G}_2(s) = \tau \mathbf{F}(\hat{\mathbf{z}}_\varphi(s), \hat{x}(s-1+\varepsilon \tilde{\hat{x}}(s)))) - \tau \mathbf{F}(\mathbf{z}_\varphi(s), x(s-1+\varepsilon \tilde{x}(s)),$$

or, using the notation introduced earlier,

$$\mathbf{G}_2(s) = \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s).$$

We now choose $q > q_1$ so that, due to (42), we have $\|\mathbf{G}_1\|_{\alpha_0} \leq \tilde{\delta}$ and therefore we may write

$$\mathbf{G}_1(t) = \tilde{\delta}\bar{\mathbf{G}}_1(t), \quad 0 \leq t \leq \alpha_0.$$

where $\|\bar{\mathbf{G}}_1\|_{\alpha_0} = 1$. From (45), we have that

$$\hat{\mathbf{z}}_\varphi - \mathbf{z}_\varphi = \delta_1(t) + \int_0^t \Big(\mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s)\Big)ds \qquad (46)$$

where

$$\delta_1(t) = \tilde{\delta}\int_0^t \bar{\mathbf{G}}_1(s)ds, \quad \text{with} \ \ \|\bar{\mathbf{G}}_1\|_{\alpha_0} \leq 1. \qquad (47)$$

From (46), we have that

$$\|\hat{\mathbf{z}}_\varphi(t) - \mathbf{z}_\varphi(t)\| \leq \|\delta_1(t)\| + \left\|\int_0^t \Big(\mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s)\Big)ds\right\|. \qquad (48)$$

Using Lemma 2, we obtain

$$\left\| \int_0^t \Big( \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s) \Big) ds \right\| \leq \frac{1}{2} \|\hat{\mathbf{z}}_\varphi - \mathbf{z}_\varphi\|, \quad 0 \leq t \leq \beta, \tag{49}$$

and therefore, combining (47), (48), and (49), and following the notation from Lemma 3, we have

$$\|\hat{\mathbf{z}}_\varphi - \mathbf{z}_\varphi\|_t \leq 2\tilde{\delta} = a_2 \tilde{\delta}, \quad 0 \leq t \leq \beta. \tag{50}$$

When $\beta < t \leq \min\{\alpha_0, 2\beta\}$, from (46) we have

$$\begin{aligned} \hat{\mathbf{z}}_\varphi(t) - \mathbf{z}_\varphi(t) &= \delta_1(t) + \int_0^\beta \Big( \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s) \Big) ds \\ &+ \int_\beta^t \Big( \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s) \Big) ds. \end{aligned}$$

Thus, for $\beta < t \leq \min\{\alpha_0, 2\beta\}$ we get

$$\|\hat{\mathbf{z}}_\varphi(t) - \mathbf{z}_\varphi(t)\| = \|\delta_1(t)\| + \left\| \int_0^\beta \Big( \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s) \Big) ds \right\| \tag{51}$$

$$+ \left\| \int_\beta^t \Big( \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s) \Big) ds \right\|. \tag{52}$$

Thanks to (49) and (50) we have

$$\left\| \int_0^\beta \Big( \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s) \Big) ds \right\| \leq a_2 \tilde{\delta}/2.$$

Then from (51) and again using the notation in Lemma 3,

$$\begin{aligned} \|\hat{\mathbf{z}}_\varphi(t) - \mathbf{z}_\varphi(t)\| \leq{}& a_1 \tilde{\delta} + a_2 \tilde{\delta}/2 + \left\| \int_\beta^t \Big( \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s) \Big) ds \right\|, \\ & \beta \leq t \leq \min\{\alpha_0, 2\beta\}. \end{aligned}$$

It then follows from Lemma 2 that

$$\left\| \int_\beta^t \Big( \mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s) - \mathbf{H}_{\varphi,\mathbf{z}}(s) \Big) ds \right\| \leq \frac{1}{2} \|\hat{\mathbf{z}}_\varphi - \mathbf{z}_\varphi\|$$

and therefore we have

$$\|\hat{\mathbf{z}}_\varphi(t) - \mathbf{z}_\varphi(t)\| \leq 2\tilde{\delta} + a_2 \tilde{\delta} = a_3 \tilde{\delta}, \quad \beta \leq t \leq \min\{\alpha_0, 2\beta\}.$$

We conclude the proof by way of induction. Assume that, for some $1 \leq j \leq L$ and for each $1 \leq k \leq j$,

$$\|\hat{\mathbf{z}}_\varphi(t) - \mathbf{z}_\varphi(t)\| \leq a_{k+1} \tilde{\delta}, \quad (k-1)\beta \leq t \leq k\beta. \tag{53}$$

Then we have

$$\begin{aligned}\|\hat{\mathbf{z}}_\varphi(t)-\mathbf{z}_\varphi(t)\| &\leq \delta_1(t)+\sum_{\ell=0}^{j-1}\left\|\int_{\ell\beta}^{(\ell+1)\beta}\Big(\mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s)-\mathbf{H}_{\varphi,\mathbf{z}}(s)\Big)ds\right\| \qquad (54)\\ &+ \left\|\int_{j\beta}^{t}\Big(\mathbf{H}_{\varphi,\hat{\mathbf{z}}}(s)-\mathbf{H}_{\varphi,\mathbf{z}}(s)\Big)ds\right\|,\\ &\qquad j\beta\leq t\leq \min\{(j+1)\beta,\alpha_0\}.\end{aligned}$$

Thanks to Lemma 2 and (54) we have

$$\begin{aligned}\|\hat{\mathbf{z}}_\varphi(t)-\mathbf{z}_\varphi(t)\| &\leq \delta_1(t)+\sum_{\ell=0}^{j-1}\frac{1}{2}\|\hat{\mathbf{z}}(s)-\mathbf{z}(s)\|_{\ell\beta\leq s\leq(\ell+1)\beta} \qquad (55)\\ &+ \frac{1}{2}\|\hat{\mathbf{z}}(t)-\mathbf{z}(t)\|, \qquad j\beta\leq t\leq \min\{(j+1)\beta,\alpha_0\},\end{aligned}$$

and thus, due to to (53), we have

$$\frac{1}{2}\|\hat{\mathbf{z}}_\varphi(t)-\mathbf{z}_\varphi(t)\| \leq \tilde{\delta}+\frac{1}{2}\sum_{\ell=0}^{j-1}a_{\ell+2}, \quad j\beta\leq t\leq \min\{(j+1)\beta,\alpha_0\},$$

i.e.,

$$\|\hat{\mathbf{z}}_\varphi(t)-\mathbf{z}_\varphi(t)\| \leq \tilde{\delta}\left(2+\sum_{\ell=0}^{j-1}a_{\ell+2}\right)=a_{j+2}\tilde{\delta}, \qquad j\beta\leq t\leq \min\{(j+1)\beta,\alpha_0\},$$

which implies that the inequality stated in (53) is valid for all $k\leq L+1$. Therefore, using Lemma 3, we deduce that

$$\|\hat{\mathbf{z}}_\varphi(t)-\mathbf{z}_\varphi(t)\|\leq 2^{(L+1)}\tilde{\delta}=\delta,$$

which ends the proof of Theorem 2.